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STABILITY AND ESTIMATES NEAR ELLIPTIC EQUILIBRIUM POINTS IN HAMILTONIAN SYSTEMS AND APPLICATIONS

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Contents

Introduction

The main subject of this thesis is the study of the nonlinear stability of elliptic equilibria in Hamiltonian systems with n degrees of freedom with $n \geq 3$. In particular, we provide a criterion to obtain formal stability that generalises previous approaches.

The kind of stability we are interested in is of formal type. To our knowledge there are no examples of systems that are formally stable but not Liapunov stable, which gives an idea of the strength of formal stability in the setting of nonlinear stability of equilibria. Moreover, we give exponential time estimates for the solutions near the equilibria of formally stable systems.

The problem of the study of the nonlinear stability of elliptic equilibria in Hamiltonian systems with $n = 2$ is already solved, but the passage to $n = 3$ is not trivial, as a whole world of possibilities arise. The study of the stability for $n = 2$ consists in the use of KAM theory. In fact, the application of Moser's twist mapping theorem allows one to prove that close to the equilibrium point there exist two-dimensional invariant tori, encasing the equilibrium. In this case the equilibrium is stable in the Liapunov sense. Otherwise, the instability is proved by using Chetaev's theorem. Unfortunately, these theorems do not allow one to draw conclusions when $n \geq 3$. In this case the existence of invariant tori, which still could be proved, does not prevent an orbit starting in a gap between tori from diffusing through the gaps, and going far from the equilibrium: this is the so-called Arnold diffusion.

There are important results in the literature for $n \neq 3$ that can be classified into two categories. The first one concerns formal stability, Lie stability being one particular case. The second one is Nekhoroshev theory and regards stability over exponentially long intervals. Up to now, both concepts have been apart. Surprisingly there is almost no connection in the literature between Nekhoroshev stability and formal stability for elliptic equilibria. Indeed, excepting the pioneering papers by Glimm and Bryuno, on the one hand the works related to formally stable systems do not consider the issue of getting estimates that measure the validity of the nonlinearly stable solutions. On the other hand there has been a significant progress in the studies on elliptic equilibria from the point of view of Nekhoroshev, obtaining very sharp bounds on the solutions but they do not deal with the existence of positive definite first integrals. The stability criterion provided in this thesis is of formal type, being Nekhoroshev stability a particular case.

Formal stability of elliptic equilibria was started by Siegel [69] and Moser [58, 59, 60], who established conditions on the quadratic terms of the Hamiltonians to achieve formal stability. Glimm [36] proved formal stability provided the quartic terms in normal form and in action-angle variables do not depend on the angles and are indeed a definite function in terms of the actions. Bryuno [12] refined previous results getting a criterion for formal stability of Hamiltonians based on the quadratic and quartic terms. Other members of the Russian school also contributed significantly to the research in formal stability starting in the decade of the 70. We quote the pioneering work by Khazin [42, 43], who established the concept of Lie stability, although he named it Birkhoff stability. In fact Birkhoff stability and Lie stability are the same for an equilibrium with semisimple linear part, but Lie stability makes sense even in the non-diagonalizable case. More papers dealing with formal stability and instability for several cases managing resonant situations are [51, 72, 45, 46]. We can also mention the contributions by dos Santos and collaborators [26, 27, 28], who prove that Lie stability implies formal stability and establish several criteria dealing with Lie stable equilibria in cases of resonances. They also treat instability using suitable Chetaev functions [21]. The instability analysis using the "invariant ray technique" is developed in [42, 43, 72].

The other philosophy to approach the study of stability relates the analysis of the asymptotic behaviour of the solutions near the equilibria. It is the so-called Nekhoroshev theory [62], where the Italian school has widely contributed [7, 8, 63, 66]. Based on Nekhoroshev theory [62] for steep functions in the setting of the stability of elliptic equilibria, several authors [35, 22, 48] established results on bounds for exponentially long times on the actual solutions near an equilibrium of an analytic Hamiltonian system. These bounds were improved later in [7, 8, 63, 66]. Recently the theory of stability has been enlarged in [68, 39] to deal with some degenerate situations where steepness is obtained from higher-order terms, and thus Nekhoroshev estimates apply. As well the papers [33, 9] deal respectively with very sharp estimates in the case of Diophantine conditions among the frequencies and the relationship between the nonlinear stability of elliptic equilibria and the existence of KAM tori nearby.

In addition to the above, in a series of papers Guzzo and coworkers, Niederman and Bounemoura have relaxed the hypotheses to get Nekhoroshev estimates, allowing the part of the Hamiltonian depending only on the actions to be non-steep. More precisely, Guzzo *et al.* [38] introduced the notion of rational convexity, which roughly means that the convexity property is tested only on the subspaces of fast drift. This idea has been generalised by Niederman [64] under the name of Diophantine steepness condition (see also an equivalent concept in [11]), which is a weak condition of transversality that involves only the affine subspaces spanned by integer vectors. This property leads to exponential estimates of stability of Nekhoroshev type. Checking these conditions on a specific problem is not usually an easy task.

In the thesis we get Lie stability under the weakest possible assumptions. We achieve it by exploiting the algebraic structure of the linear part of the equation as much as we can. We do not need to check whether the truncated normal form Hamiltonian vanishes for all non-null vectors of the orthogonal space related to the frequency vector, but only for a subspace of it. This allows us to obtain Lie stable systems for which exponential time estimates apply but such that they do not satisfy the conditions of Nekhoroshev estimates appearing in [38, 64, 11].

The instability issue is dealt with in this thesis by means of building a suitable invariant ray. This is accomplished for n-degrees of freedom Hamiltonians under the existence of a single resonance as well as for Hamiltonians with a multiple resonance of order odd, simplifying previous results. In both situations Chetaev's functions need to be obtained.

We apply our results to study the stability (and instability) of specific equilibrium solutions in two interesting problems of Celestial Mechanics, namely, the Lagrangian points L_4 and L_5 of the spatial restricted circular three-body problem and an equilibrium point in a problem of an artificial satellite moving in a circular orbit around its centre of mass. The satellite is considered a rigid body. In both systems we perform a deep analysis in terms of the corresponding parameters, identifying the unstable situations and enlarging previous results on the same problems. Moreover on the Lie stable cases we provide estimates over exponentially long times.

Chapter 1

Preliminaries

In this chapter we introduce some basic concepts about the study of Hamiltonian dynamic systems. Specifically, linear, nonlinear, formal stability, exponential time estimates and perturbation techniques.

1.1 Perturbation theory

We introduce some basic concepts and result related to Averaging Theory for Hamiltonian systems and KAM Theory. The use of symplectic transformations to simplify a Hamiltonian system has been employed widely in problems related to stability theory and Celestial Mechanics. Here we summarise some well known concepts.

1.1.1 Lie transformations

The method of Lie transformations, initiated by Deprit [24], is a procedure to define a near-identity symplectic change of variables in a system of equations that depends on a small parameter. We introduce Lie transformations following the book by Meyer, Hall, Offin [53].

Definition 1.1. *A symplectic change of variables* $\mathbf{x} \equiv \mathbf{X}(\mathbf{y};\varepsilon)$ *is called near-identity if it is symplectic for each fixed* ε *and is of the form* $\mathbf{X}(\mathbf{y};\varepsilon) = \mathbf{y} + \mathcal{O}(\varepsilon)$*; i.e.,* $\mathbf{X}(\mathbf{y};0) = \mathbf{y}$.

Because $\mathbf{X}(y,0) = 0$ and $\partial\mathbf{X}(y,\varepsilon)/\partial y$ is nonsingular for small ε, thus by the inverse function theorem, the map $y \to \mathbf{X}(y,\varepsilon)$ has a differentiable inverse. In consequence, if $\mathbf{y} \equiv \mathbf{Y}(\mathbf{X}(\mathbf{y};\varepsilon);\varepsilon)$ is the inverse of $\mathbf{x} \equiv \mathbf{X}(\mathbf{Y}(\mathbf{x};\varepsilon);\varepsilon)$, then both are symplectic for fixed ε.

Theorem 1.2. *The transformation* $\mathbf{X}(\mathbf{y};\varepsilon)$ *is a near-identity symplectic change of variables if and only if it is a general solution of a Hamiltonian system of the form*

$$\frac{d\mathbf{x}}{d\varepsilon} = \mathbb{J}\nabla\mathcal{W}(\mathbf{x};\varepsilon), \quad \mathbf{x}(0) = \mathbf{y}$$

where $\mathcal{W}$ *is smooth and* $\mathbb{J}$ *is the usual skew-symmetric matrix.*

See the proof in [53].

Let $\mathcal{H}(\mathbf{x};\varepsilon)$ be a Hamiltonian and $\mathcal{G}(\mathbf{y};\varepsilon) \equiv \mathcal{H}(\mathbf{X}(\mathbf{y};\varepsilon);\varepsilon)$ the Hamiltonian in the new coordinates. $\mathcal{G}$ is called the Lie transformation of $\mathcal{H}$ generated by $\mathcal{W}$. The method of *Lie transformations* is introduced by the following formulas

$$\mathcal{H}(\mathbf{x};\varepsilon) = \mathcal{H}_*(\mathbf{x};\varepsilon) = \sum_{i=0}^{\infty} \frac{\varepsilon^i}{i!}\mathcal{H}_i^0(\mathbf{x}), \tag{1.1}$$

$$\mathcal{G}(\mathbf{y};\varepsilon) = \mathcal{H}^*(\mathbf{y};\varepsilon) = \sum_{i=0}^{\infty} \frac{\varepsilon^i}{i!}\mathcal{H}_0^i(\mathbf{y}), \tag{1.2}$$

$$\mathcal{W}(\mathbf{x};\varepsilon) = \sum_{i=0}^{\infty} \frac{\varepsilon^i}{i!}\mathcal{W}_{i+1}(\mathbf{x}), \tag{1.3}$$

where $\{\mathcal{H}_j^i\}$ for $i = 1, 2, \dots$ and $j = 0, 1, \dots$ satisfy the recursive identities

$$\mathcal{H}_j^i = \mathcal{H}_{j+1}^{i-1} + \sum_{k=0}^{j} \binom{j}{k} \{\mathcal{H}_{j+k}^{i-1}, \mathcal{W}_{k+1}\}.$$

The relationship among these functions is illustrated more easily in the Lie triangle

$$\begin{array}{ccccc}
\mathcal{H}_0^0 & & & & \\
\downarrow & & & & \\
\mathcal{H}_1^0 & \rightarrow & \mathcal{H}_0^1 & & \\
\downarrow & & \downarrow & & \\
\mathcal{H}_2^0 & \rightarrow & \mathcal{H}_1^1 & \rightarrow & \mathcal{H}_0^2 \\
\downarrow & & \downarrow & & \downarrow
\end{array}$$

For example, to compute the series expansion for $\mathcal{H}_*$ through terms of order ε^2, one first determines $\mathcal{H}_0^1$ by the formula

$$\mathcal{H}_0^1 = \mathcal{H}_1^0 + \{\mathcal{H}_0^0, \mathcal{W}_1\}$$

which gives the term of order ε and then one computes

$$\mathcal{H}_1^1 = \mathcal{H}_2^0 + \{\mathcal{H}_1^0, \mathcal{W}_1\} + \{\mathcal{H}_0^0, \mathcal{W}_2\}$$

and $\mathcal{H}_0^2 = \mathcal{H}_1^1 + \{\mathcal{H}_0^1, \mathcal{W}_1\}$ getting

$$\mathcal{H}^*(\mathbf{x};\varepsilon) = \mathcal{H}_0^0(\mathbf{x}) + \varepsilon\mathcal{H}_0^1(\mathbf{x}) + \frac{\varepsilon^2}{2}\mathcal{H}_0^2(\mathbf{x}) + \cdots$$

1.1.2 Averaging and normal forms

Let (M,Ω) be a symplectic manifold of dimension $2n$, $\mathcal{H}_0\colon M \to \mathbb{R}$ a smooth Hamiltonian which defines a Hamiltonian vector field $Y_0 = (dH_0)^{\#}$ with symplectic flow φ_0^t. Let $I \subset \mathbb{R}$ be an interval such that each $h \in I$ is a regular value of $\mathcal{H}_0$ and $\mathcal{N}_0(h) = \mathcal{H}_0^{-1}(h)$ is a compact connected circle bundle over a base space $\mathcal{B}(h)$ with projection $\pi\colon \mathcal{N}_0(h) \to \mathcal{B}(h)$. So, this is the setting of regular reduction theory. Assume that all the solutions of Y_0 in $\mathcal{N}_0(h)$ are periodic and have periods smoothly depending only on the value of the Hamiltonian; i.e., the period is a smooth function $T = T(h)$.

Let ε be a small parameter, $\mathcal{H}_1\colon M \to \mathbb{R}$ be smooth, $\mathcal{H}_\varepsilon = \mathcal{H}_0 + \varepsilon\mathcal{H}_1$, $Y_\varepsilon = Y_0 + \varepsilon Y_1 = d\mathcal{H}_\varepsilon^{\#}$, $\mathcal{N}_\varepsilon(h) = \mathcal{H}_\varepsilon^{-1}(h)$, $\pi\colon \mathcal{N}_\varepsilon(h) \to \mathcal{B}(h)$ the projection, and ϕ_ε^t be the flow defined by Y_ε.

Let the average of $\mathcal{H}_1$ be

$$\bar{\mathcal{H}} = \frac{1}{T}\int_0^T \mathcal{H}_1(\phi_0^t)dt. \tag{1.4}$$

The next result provides sufficient conditions for characterising the existence of periodic solutions of the Hamiltonian system associated to $\mathcal{H}_\varepsilon$. For more information on this subject the reader is addressed to [67], [31] and [56].

Theorem 1.3 (Reeb)**.** *If $\bar{\mathcal{H}}$ has a non-degenerate critical point at $\pi(p) = \bar{p} \in \mathcal{B}(h)$ with $p \in \mathcal{N}_0(h)$, then there are smooth functions $p(\varepsilon)$ and $T(\varepsilon)$ for ε small with $p(0) = p$, $T(0) = T$, and $p(\varepsilon) \in \mathcal{N}_\varepsilon$, and the solution of Y_ε through $p(\varepsilon)$ is $T(\varepsilon)$-periodic. In addition, if the characteristic exponents of the critical point $\bar{p}$ (that is, the eigenvalues of the matrix $A = \mathbb{J}D^2\bar{\mathcal{H}}(\bar{p})$) are $\lambda_1, \lambda_2, \ldots, \lambda_{2n-2}$, then the characteristic multipliers of the periodic solution through $p(\varepsilon)$ are*

$$1, 1, 1 + \varepsilon\lambda_1 T + O(\varepsilon^2), 1 + \varepsilon\lambda_2 T + O(\varepsilon^2), \cdots, 1 + \varepsilon\lambda_{2n-2}T + O(\varepsilon^2).$$

Theorem 1.4. *Let p and $\bar{p}$ as in the previous Theorem. If one or more of the characteristic exponents λ_j is real or has nonzero real part, then the periodic solution through $p(\varepsilon)$ is unstable. If the matrix A is strongly stable, then the periodic solution through $p(\varepsilon)$ is elliptic, i.e., linearly stable.*

The proofs of Theorems 1.3 and 1.4 appear in [31].

For the case in which h is a regular value but the energy level $\mathcal{N}_0(h)$ is not compact, a recent result due a Meyer, Palacián and Yanguas [**?**] show that the previous theorems also are valid.

The essence of normalization is to use Lie transformations to simplify a Hamiltonian system [24]. When the Hamiltonian, and hence the equations, are in sufficiently simple form, they are said to be in "normal form". In this way, if $\mathbf{x} \equiv \mathbf{X}(\mathbf{y};\varepsilon)$ is a Lie transformation such that the transformed Hamiltonian $\mathcal{K}(\mathbf{y};\varepsilon) = \mathcal{H}(\mathbf{x};\varepsilon)$ is in its normal form. Then, $\mathcal{K}$ is called the normal form of $\mathcal{H}$.

Given an analytic Hamiltonian $\mathcal{H}$ which has an equilibrium point at the origin in $\mathbb{R}^{2n}$ and is zero at the origin, then $\mathcal{H}$ can be expanded in Taylor series by

$$\mathcal{H}(\mathbf{x};\varepsilon) = \sum_{i=0}^{\infty} \frac{\varepsilon^i}{i!}\mathcal{H}_i^0(\mathbf{x}), \tag{1.5}$$

where $\mathcal{H}_i^0$ is a homogeneous polynomial in $\mathbf{x}$ of degree $i+2$. The linearised equations about the critical point $\mathbf{x} = \mathbf{0}$ are

$$\dot{\mathbf{x}} = A\mathbf{x} = \mathbb{J}S\mathbf{x} = \mathbb{J}\nabla\mathcal{H}_0^0,$$

where S is a $2n \times 2n$ real symmetric matrix, and $A = \mathbb{J}S$ is a Hamiltonian matrix. The general solution of the linearised system is $\varphi(t,\xi) = \exp(At)\xi$.

The general theorem on normal forms is as follows.

Theorem 1.5. *Let A be a Hamiltonian matrix. Then there exists a formal symplectic change of variables, $\mathbf{x} = \mathbf{X}(\mathbf{y};\varepsilon) = \mathbf{y} + \dots$, that transforms the Hamiltonian (1.5) to*

$$\mathcal{H}(\mathbf{y};\varepsilon) = \sum_{j=0}^{\infty} \frac{\varepsilon^j}{j!}\mathcal{H}_0^j(\mathbf{y}),$$

where $\mathcal{H}_0^j$ is a homogeneous polynomial of degree $j+2$ such that

$$\mathcal{H}_0^j(e^{A^T t}\mathbf{y}) \equiv \mathcal{H}_0^j(\mathbf{y}), \tag{1.6}$$

for all $j = 0, 1, \dots$, all $\mathbf{y} \in \mathbb{R}^{2n}$, and all $t \in \mathbb{R}$.

Let $\mathcal{H}_0^T = \frac{1}{2}\mathbf{x}^T R\mathbf{x}$ be the quadratic Hamiltonian for the adjoint linear equation; so, $A^T = \mathbb{J}R$. Then (1.6) is equivalent to $\left\{\mathcal{H}^i, \mathcal{H}_0^T\right\} = 0$ for all i.

For the case in which the matrix A is semisimple, that is, diagonalisable over the complex numbers, we have the following result.

Theorem 1.6. *Let A be semisimple. Then there exists a formal symplectic change of variables, $\mathbf{x} = \mathbf{X}(\mathbf{y};\varepsilon) = \mathbf{y} + \dots$, that transforms the Hamiltonian (1.5) to*

$$\mathcal{H}(\mathbf{y};\varepsilon) = \sum_{i=0}^{\infty} \frac{\varepsilon^i}{i!}\mathcal{H}_0^i(\mathbf{y}), \tag{1.7}$$

where $\mathcal{H}^i$ is a homogeneous polynomial of degree $i+2$ such that

$$\mathcal{H}_0^i(e^{At}\mathbf{y}) \equiv \mathcal{H}_0^i(\mathbf{y}) \tag{1.8}$$

for all $i = 0, 1, \dots,$ all $\mathbf{y} \in \mathbb{R}^{2n}$, and all $t \in \mathbb{R}$.

Formula (1.8) is the classical definition of normal form for a Hamiltonian near an equilibrium point with a semisimple linear part, and is equivalent to $\{\mathcal{H}_0^i, \mathcal{H}_0^0\} = 0$ for all i.

1.1.3 KAM theory

We want to study the dynamics of a Hamiltonian system with respect to the influence of small Hamiltonian perturbations. This is achieved by applying KAM theory, [6]. The classical KAM theory demands two properties of the unperturbed system, namely, the integrability and the non-degeneracy.

Considering perturbed integrable Hamiltonian systems of the form

$$\mathcal{H}(I,\varphi,\varepsilon) \;=\; \mathcal{H}_0(I) + \varepsilon\mathcal{H}_1(I,\varphi,\varepsilon), \tag{1.9}$$

where ε is a small parameter. The phase space associated to $\mathcal{H}_0$ is foliated by invariant tori and there are n independent first integrals of motion. That is to say, a level set of the n independent first integrals of motion is diffeomorphic to an n-dimensional torus $T^n = \{\varphi = (\varphi_1, \dots, \varphi_n) \mod 2\pi\}$, φ_i being angular coordinates for $i = 1, \dots, n$. The frequencies of the motions are given by $\omega_i = d\varphi_i/dt$. In order to maintain the Hamiltonian structure, action coordinates – $I = (I_1, \dots, I_n)$ – are defined and together with the angles define the phase space of the system and are called action-angle variables. Action coordinates are related with the frequencies by $\omega_i = \partial\mathcal{H}_0/\partial I_i$ and the trajectories describing these motions are dense in the tori. These motions are known by quasi-periodic motions.

A system is non-degenerate if the determinant $|\partial^2\mathcal{H}_0/\partial I^2| = |\partial\dot{\varphi}/\partial I|$ is not zero in an open domain of the phase space. It means that the frequencies are functionally independent.

Definition 1.7. *The frequencies $\omega = (\omega_1, \dots, \omega_n)$ are called resonant if they are rationally independent, i.e.*

$$k \cdot \omega \neq 0 \;\; \textit{for all} \;\; k \in \mathbb{Z}^n \setminus \{0\},$$

and are non-resonant otherwise.

In the non-resonant case, each orbit is dense on the n-torus and in the resonant case, the torus decomposes into an m-parameter family of invariant $(n-m)$-tori and given an orbit it is dense on a lower-dimensional torus.

Kolmogorov [1], Arnold [5] and Moser [61] proved the persistence of those tori, whose frequencies verify the *Diophantine condition*.

If we ask about the existence of these Diophantine frequencies, this is answered with:

Lemma 1.8. *(Arnold) Let $\Omega \in \mathbb{R}^n$ be a bounded domain and let $\tau > n-1$ be fixed. Almost all vectors $\omega \in \Omega$ satisfy the Diophantine condition.*

The classical KAM theorem states this fact in the following way.

Theorem 1.9. *(Kolmogorov, Arnold and Moser) Consider the system of equations induced by an analytic Hamiltonian $\mathcal{H}_0$ to be non-degenerate, then most of the invariant tori which exist for the unperturbed system $(\varepsilon = 0)$ will, slightly deformed, also exist for $\varepsilon \neq 0$ sufficiently small. Moreover, the Lebesgue measure of the complement of the set of tori tends to zero as ε tends to zero.*

There is a variation of the KAM theorem for isoenergetically non-degenerate systems.

Definition 1.10. *An n-dimensional system is isoenergetically non-degenerate if*

$$\begin{vmatrix} \dfrac{\partial^2 \mathcal{H}_0}{\partial I^2} & \dfrac{\partial \mathcal{H}_0}{\partial I} \\ \dfrac{\partial \mathcal{H}_0}{\partial I} & 0 \end{vmatrix} \neq 0.$$

Theorem 1.11 (Kolmogorov)**.** *If $\mathcal{H}_0$ is non-degenerate or isoenergetically non-degenerate, then under a sufficiently small Hamiltonian perturbation most of the non-resonant invariant tori do not disappear but are only slightly deformed, so that in phase space of the perturbed system there also exist invariant tori. In the case of isoenergetic non-degeneracy the invariant tori form a majority on each energy-level manifold.*

There are systems where $\mathcal{H}_0$ does not depend on all the actions, they are the so called properly degenerate. In this case, the perturbation is said to *remove the degeneracy* if the full Hamiltonian can be written as

$$\mathcal{H}(I, \varphi, \varepsilon) \;=\; \mathcal{H}_{00}(I) + \varepsilon \mathcal{H}_{01}(I) + \varepsilon^2 \mathcal{H}_{11}(I, \varphi, \varepsilon), \qquad (1.10)$$

where $\mathcal{H}_{00}$ depends only on the first r action variables and is either non-degenerate or isoenergetically non-degenerate with respect to these variables and $\mathcal{H}_{01}$ is non-degenerate with respect to the last $n-r$.

Theorem 1.12 (Arnold)**.** *Suppose that the unperturbed system is degenerate, but the perturbation removes the degeneracy. Then a larger part of the phase space is filled with invariant tori that are close to the invariant tori $I = const$ of the intermediate system. Among these frequencies, r correspond to the fast phases, and $n - r$ to the slow phases. If the unperturbed Hamiltonian is isoenergetically non-degenerate with respect to those r variables on which it depends, then the invariant tori just described form a majority on each energy-level manifold of the perturbed system.*

Sometimes one can detect invariant tori using KAM Theory and at times even stability. The following result due to Dumas, Meyer, Palacián and Yanguas [57] applies for Hamiltonians systems with two degrees of freedom.

We consider the Hamiltonian

$$\mathcal{H}_\varepsilon(I, \theta, y) = \mathcal{H}_0(I) + \varepsilon \mathcal{H}_1(I, \theta, y) = \mathcal{H}_0(I) + \varepsilon \bar{\mathcal{H}}(I, y) + O(\varepsilon),$$

where $\bar{\mathcal{H}}$ is defined as in (1.4).

Theorem 1.13 (Meyer-Palacián-Yanguas)**.** *Let $n = 2$ and let p be as in Theorem 1.3. Suppose there are symplectic action-angle variables (I_1, θ_1) at $\bar{p}$ in $\mathcal{B}(h)$ such that*

$$\bar{\mathcal{H}} = \omega_1 I_1 + \varepsilon \mathcal{K}(I, I_1) + O(\varepsilon^2),$$

where ω_1 is non-zero and

$$\frac{\partial \mathcal{K}(I, I_1)}{\partial I_1^2} \neq 0.$$

Then for suficiently small $\varepsilon > 0$ encircling the periodic solutions given in Theorem 1.3 there are invariant KAM tori of dimension 2. In particular the periodic solutions are orbitally stable.

There are many other results on KAM theory such as Moser's invariant curve Theorem, Arnold's stability Theorem for two degrees-of-freedom Hamiltonians and others, as well as many related results, see for instance [6].

There are cases in which the previous theorems cannot be applied on our context. Thus, we apply Han, Li and Yi's Theorem, designed specifically to deal with highly degenerated Hamiltonians, which turns to be essential to obtain the results in Chapters 2 and 3. This theorem is introduced as follows.

Consider a Hamiltonian system of the form

$$\mathcal{H}_\varepsilon(I, \varphi, \varepsilon) = h_0(I^{n_0}) + \varepsilon^{m_1} h_1(I^{n_1}) + \cdots + \varepsilon^{m_a} h_a(I^{n_a}) + \varepsilon^{m_a+1} p(I, \varphi, \varepsilon), \quad (1.11)$$

where $(I, \varphi) \in \mathbb{R}^n \times \mathbb{T}^n$ are action-angle variables with the standard symplectic structure $dI \wedge d\varphi$, and $\varepsilon > 0$ is a sufficiently small parameter. Hamiltonian $\mathcal{H}_\varepsilon$ is real analytic, and the parameters a, m, n_i $(i = 0, 1, \dots, a)$ and m_j $(j = 1, 2, \dots, a)$ are positive integers satisfying $n_0 \le n_1 \le \dots \le n_a = n$, $m_1 \le m_2 \le \dots \le m_a = m$, $I^{n_i} = (I_1, \dots, I_{n_i})$, for $i = 1, 2, \dots, a$, and p depends on ε smoothly.

The Hamiltonian $\mathcal{H}_\varepsilon(I, \varphi, \varepsilon)$ is taken in a bounded closed region $Z \times \mathbb{T}^n \subset \mathbb{R}^n \times \mathbb{T}^n$. For each ε the integrable part of $\mathcal{H}_\varepsilon$,

$$X_\varepsilon(I) = h_0(I^{n_0}) + \varepsilon^{m_1} h_1(I^{n_1}) + \dots + \varepsilon^{m_a} h_a(I^{n_a}),$$

admits a family of invariant n-tori $T^\varepsilon_\zeta = \{\zeta\} \times \mathbb{T}^n$, with linear flows $\{x_0 + \omega^\varepsilon(\zeta)t\}$, where, for each $\zeta \in Z$, $\omega^\varepsilon(\zeta) = \nabla X_\varepsilon(\zeta)$ is the frequency vector of the n-torus T^ε_ζ and ∇ is the gradient operator. When $\omega^\varepsilon(\zeta)$ is nonresonant, the n-torus T^ε_ζ becomes quasi-periodic with slow and fast frequencies of different scales. We refer to the integrable part X_ε and its associated tori $\{T^\varepsilon_\zeta\}$ as the intermediate Hamiltonian and intermediate tori, respectively.

Let $\bar{I}^{n_i} = (I^{n_{i-1}+1}, \dots, I^{n_i})$, $i = 0, 1, \dots, a$ (where $n_{-1} = 0$, hence $\bar{I}^{n_0} = I^{n_0}$), and define

$$\Omega = \left(\nabla_{\bar{I}^{n_0}} h_0(I^{n_0}), \dots, \nabla_{\bar{I}^{n_a}} h_{n_a}(I^{n_a})\right),$$

such that, for each $i = 0, 1, \dots, a$, $\nabla_{\bar{I}^{n_i}}$ denotes the gradient with respect to $\bar{I}^{n_i}$.

We assume the following high-order degeneracy-removing condition of Bruno-Rüssman type (so named by Han, Li, and Yi), giving credit to Bruno and Rüssman, who provided weak conditions on the frequencies guaranteeing the persistence of invariant tori, the so-called (A) condition: there is a positive integer s such that

$$Rank\{\partial^\alpha \Omega(I) \,:\, 0 \le |\alpha| \le s\} = n, \quad \forall\, I \in Z.$$

For the usual case of a nearly integrable Hamiltonian system of the type

$$\mathcal{H}_\varepsilon(I, \varphi, \varepsilon) = X(I) + \varepsilon p(I, \varphi, \varepsilon), \quad (I, \varphi) \in Z \times \mathbb{T}^n \subset \mathbb{R}^n \times \mathbb{T}^n. \tag{1.12}$$

Condition (A) given above generalises the classical Kolmogorov non-degenerate condition that $\partial\Omega(I)$ be nonsingular over Z, where $\Omega(I) = \nabla X(I)$; Bruno's non-degenerate condition that $Rank\{\Omega(I), \partial\Omega\} = n$, $\forall\, I \in Z$; and the weakest non-degenerate condition guaranteeing such persistence provided by Rüssman, that $\omega(Z)$ should not lie in any $(n-1)$-dimensional subspace. Rüssman condition is equivalent to condition (A) for systems like (1.12). However, Bruno or Rüssman conditions do not apply to Hamiltonian (1.11), as it is too degenerate.

The following theorem gives the right setting in which one can ensure the persistence of KAM tori for Hamiltonian like (1.11).

Theorem 1.14 (Han, Li and Yi). *Assume the condition (A), and let δ with $0 < \delta < 1/5$ be given. Then there exists an $\varepsilon_0 > 0$ and a family of Cantor sets $Z_\varepsilon \subset Z$, $0 < \varepsilon < \varepsilon_0$, with $|Z \setminus Z_\varepsilon| = O(\varepsilon^{\delta/s})$, such that each $\zeta \in Z_\varepsilon$ corresponds to a* *real analytic, invariant, quasi-periodic n-torus $\bar{T}^{\varepsilon}_{\zeta}$ of Hamiltonian (1.11), which is slightly deformed from the intermediate n-torus T^{ε}_{ζ}. Moreover, the family $\{\bar{T}^{\varepsilon}_{\zeta} : \zeta \in Z_{\varepsilon},\ 0 < \varepsilon < \varepsilon_0\}$ varies Whitney smoothly.*

See the proof in [41].

1.2 Stability in Hamiltonian systems

In this section we introduce the concepts of Lie stability, Liapunov stability, formal stability of autonomous Hamiltonian systems. In addition, we recall the results of linear and non-linear stability, as well as stability results in two degrees of freedom and n degrees of freedom for autonomous Hamiltonian systems. Finally, we put several versions of Chetaev's theorem of instability of autonomous Hamiltonian systems.

1.2.1 Linear Hamiltonian systems

Consider the linear Hamiltonian system

$$\dot{z} = Az = \mathbf{J}\nabla\mathbb{H}(z), \quad \mathcal{H}_0 = \tfrac{1}{2}z^T S z \tag{1.13}$$

where S is a $2n \times 2n$ real symmetric matrix, $\mathbb{J}$ is the standard $2n \times 2n$ symplectic matrix of Hamiltonian theory and $A = \mathbb{J}S$ is a Hamiltonian matrix.

Definition 1.15. *The system of linear Hamiltonian equations (1.13) is stable if all solutions of (1.13) are bounded for all $t \in \mathbb{R}$, i.e. $\left\|e^{At}\right\|$ is uniformly bounded. Equivalently (1.13) is stable if A satisfies the pure imaginary-diagonalizable condition, PIDC, i.e. if all the eigenvalues of A are pure imaginary and A is diagonalizable over the complex numbers.*

Thus, if A satisfies the PIDC then one can choose real symplectic coordinates $(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^{2n}$ such that

$$\mathbb{H} = \frac{\omega_1}{2}(x_1^2 + y_1^2) + \frac{\omega_2}{2}(x_2^2 + y_2^2) + \cdots + \frac{\omega_n}{2}(x_n^2 + y_n^2),$$

where the eigenvalues of A are $\pm\omega_1 i, \pm\omega_2 i, \ldots, \pm\omega_n i$. In the same way, one can choose action-angle variables

$$I_j = \frac{1}{2}(x_j^2 + y_j^2), \quad \phi_j = \arctan\left(\frac{y_j}{x_j}\right), \quad \text{for } j = 1, 2, \ldots, n \tag{1.14}$$

such that

$$\mathbb{H} = \omega_1 I_1 + \omega_2 I_2 + \cdots + \omega_n I_n. \tag{1.15}$$

Now, if S is a definite matrix or equivalently all the ω_j's are of the same sign, then Dirichlet's Theorem [25] implies that the system (1.13) is stable and also implies that a small linear Hamiltonian perturbation is stable. This leads to the following concept and theorem.

Definition 1.16. *The linear Hamiltonian system (1.13) is parametrically stable or strongly stable if it and all sufficiently small linear Hamiltonian perturbations of it are stable. That is, (1.13) is parametrically stable if there is an $\varepsilon > 0$ such that $\dot{z} = Bz$ is stable, where B is any linear Hamiltonian matrix with $\|B - A\| < \varepsilon$.*

Let $\pm\alpha_1 i, \pm\alpha_2 i, \ldots, \pm\alpha_s i$ be the eigenvalues of the matrix A, and let $\mathbb{V}_j$, $j = 1, \ldots, s$, be the maximal real linear subspace where A has eigenvalues $\pm\alpha_j i$. So $\mathbb{V}_j$ is an A-invariant symplectic subspace, A restricted to $\mathbb{V}_j$ has eigenvalues $\pm\alpha_j i$, and $\mathbb{R}^{2n} = \mathbb{V}_1 \oplus \mathbb{V}_2 \oplus \cdots \oplus \mathbb{V}_s$. Let $\mathbb{H}_j$ be the restriction of $\mathbb{H}$ to $\mathbb{V}_j$.

Definition 1.17. *We will say that the linear Hamiltonian system (1.13) satisfies the Krein-Gel'fand-Lidskii condition, KGLC, if A is nonsingular, A is stable, and the Hamiltonian $\mathbb{H}_j$ is positive or negative definite for each $j = 1, \ldots, s$.*

Theorem 1.18. *The linear Hamiltonian system (1.13) is parametrically stable, if and only if, KGLC holds.*

See the proof in [54].

Group the eigenvalues of A into r groups as follows:

$$\begin{array}{cccc}
\pm\omega_1 k_{11} i, & \pm\omega_1 k_{12} i, & \ldots, & \pm\omega_1 k_{1s_1} i \\
\pm\omega_2 k_{21} i, & \pm\omega_2 k_{22} i, & \ldots, & \pm\omega_2 k_{2s_2} i \\
\vdots & \vdots & & \vdots \\
\pm\omega_r k_{r1} i, & \pm\omega_r k_{r2} i, & \ldots, & \pm\omega_r k_{rs_r} i
\end{array}$$

where $\omega_1, \ldots, \omega_r$ are rationally independent and $k_{11} \ldots k_{rs_r}$ are nonzero integers. Let $\mathbb{W}_j = [\eta(\omega_j k_{j1} i) \oplus (-\omega_j k_{j1} i)] \oplus \ldots \oplus [\eta(\omega_j k_{j\sigma} i) \oplus \eta(-\omega_j k_{j\sigma} i)]$. Here we write σ for s_j to avoid double subscripts. Again $\mathbb{W}_j$ satisfies the reality condition that $w \in \mathbb{W}$ if and only if $\bar{w} \in \mathbb{W}$, so it is the complexification of a real A-invariant symplectic subspace $\mathbb{V}_j$ and

$$\mathbb{R}^{2n} = \mathbb{V}_1 \oplus \mathbb{V}_2 \oplus \ldots \oplus \mathbb{V}_r.$$

Thus, if A_j be the restriction of A to the subspace $\mathbb{V}_j$ and $\mathbb{H}_j$ be the restriction of $\mathbb{H}$ to $\mathbb{V}_j$, then A_j has eigenvalues

$$\pm\omega_j k_{j1} i,\ \pm\omega_j k_{j2} i,\ \ldots \pm \omega_j k_{j\sigma} i.$$

Definition 1.19. *We will say that the linear Hamiltonian (1.13) satisfies the Moser-Weinstein condition, MWC, if each $\mathbb{H}_j$ is either positive or negative definite.*

If we write the Hamiltonian $\mathbb{H}$ in the form

$$\mathbb{H} = \omega_1(k_{11}I_{11} + \cdots + k_{1s}I_{1s_1}) + \cdots + \omega_r(k_{r1}I_{r1} + \cdots + k_{rs_r}I_{rs_r})$$

then the linear Hamiltonian system (1.13) satisfies MCV if and only if all the $k_{\alpha\beta}$ can be chosen as positive integers and then

$$\mathbb{H}_j = \omega_j(k_{j1}I_{j1} + k_{j2}I_{j2} + \cdots + k_{js_j}I_{j\sigma})$$

is positive or negative definite as ω_j is positive or negative. Note that MWC is stronger than KGLC.

1.2.2 Nonlinear Hamiltonian systems

Consider the autonomous Hamiltonian system with n degrees of freedom

$$\dot{x} = \mathcal{J}\nabla H(x), \tag{1.16}$$

such that the origin of the phase space is an equilibrium solution, $\mathcal{J}$ is the standard $2n \times 2n$ symplectic matrix of Hamiltonian theory [53] and $H = H(x)$ is a real analytic function of $x = (x_1, \ldots, x_n, y_1, \ldots, y_n)$. It is assumed that the Taylor series of H in a neighborhood of the origin is

$$H = H_2 + H_3 + \cdots + H_j + \cdots, \tag{1.17}$$

where H_j represents a homogeneous polynomial of degree j in x, that is,

$$H_j = \sum_{|k|+|l|=j} h_{kl}\, x^k y^l, \tag{1.18}$$

with $k = (k_1, \ldots, k_n) \in \mathbb{Z}^n$, $l = (l_1, \ldots, l_n) \in \mathbb{Z}^n$, $|k| = |k_1| + \cdots + |k_n|$, $|l| = |l_1| + \cdots + |l_n|$, $h_{kl} = h_{k_1\ldots k_n l_1\ldots l_n}$, $x^k = x_1^{k_1} \cdots x_n^{k_n}$ and $y^l = y_1^{l_1} \cdots y_n^{l_n}$. The term H_2 represents the quadratic Hamiltonian

$$H_2(x) = \tfrac{1}{2} x^T B x, \tag{1.19}$$

with $B = B^T$ a $2n \times 2n$ real symmetric matrix. The linearized equations of motion are

$$\dot{x} = Ax, \quad A = \mathcal{J}B, \tag{1.20}$$

where A is a $2n \times 2n$ real Hamiltonian matrix. In the thesis A is nonsingular and the linearized system is stable, i.e., all the eigenvalues of A are nonzero purely imaginary numbers, say $\pm\omega_1 i, \ldots, \pm\omega_n i$ and A is diagonalizable over the complex numbers. It is assumed that the non-degenerate equilibrium solution of the Hamiltonian system (3.1) is stable in the linear approximation. We can suppose, without loss of generality (see [51], [59] for more details), that a linear canonical transformation has already been constructed such that

$$H_2 = \frac{\omega_1}{2}(x_1^2 + y_1^2) + \cdots + \frac{\omega_n}{2}(x_n^2 + y_n^2).$$

In the nonsingular case, possibly after making a suitable linear symplectic transformation to bring the linear part to diagonal form, customarily one can introduce action-angle variables $I = (I_1, \ldots, I_n)$, $\theta = (\theta_1, \ldots, \theta_n)$ such that

$$I_j = \tfrac{1}{2}(x_j^2 + y_j^2)\,, \quad \theta_j = \tan^{-1}\frac{y_j}{x_j}\,,$$

and H_2 takes the form

$$H_2 = \omega_1 I_1 + \cdots + \omega_n I_n. \tag{1.21}$$

In [35, 22] strong nonresonance conditions are imposed on the eigenvalues which we do not require. We assume that H_2 is an indefinite quadratic form in terms of x, in other words, the signs of the ω_i are not all the same.

The *normal form Hamiltonian* of H defined in (2.2) up to a finite order p is the function

$$H = H_2 + \mathcal{H}_3 + \cdots + \mathcal{H}_p + \cdots \tag{1.22}$$

obtained from (2.2) through a symplectic change of coordinates whose series expansion in x starts at degree two, such that each term $\mathcal{H}_j$ is a homogeneous polynomial of degree j in x, and satisfies $\{\mathcal{H}_j, H_2\} = 0$, $j = 2, \ldots, p$, see for instance [53]. During all this work $\mathcal{H}^p$ represents the truncation of the Hamiltonian function of order p, that is,

$$\mathcal{H}^p = H_2 + \mathcal{H}_3 + \cdots + \mathcal{H}_p,$$

with $\mathcal{H}_j$ $(j = 2, \ldots, m)$ defined in (3.3). One can express the Hamiltonians $\mathcal{H}_j$ in terms of I and θ because they satisfy d'Alembert condition, see [53]. Thus we write down the normal form Hamiltonian as

$$H(I, \theta) = H_2(I) + \cdots + \mathcal{H}_{2l-2}(I) + \mathcal{H}_m(I, \theta) + \cdots \tag{1.23}$$

with $l \geq 2$, $m = 2l - 1$ or $m = 2l$ and $m \leq p$. Notice that H is an analytic function of the variables $I_j^{1/2}$, θ_j and is 2π-periodic in θ_j, $j = 1, \ldots, n$.

We recall the notion of resonance vector.

Definition 1.20. *The system (2.1) presents a resonance relation if there exists an integer vector $k = (k_1, \ldots, k_n) \neq 0$ such that*

$$k_1\omega_1 + \cdots + k_n\omega_n = 0.$$

The number $|k| = |k_1| + \cdots + |k_n|$ is called the order of the resonance. On the other hand, if

$$k_1\omega_1 + \cdots + k_n\omega_n \neq 0$$

holds for all integer vectors $k = (k_1, \ldots, k_n) \in \mathbb{Z}^n$ satisfying the equalities $|k| = j$, for $j = 1, \ldots, s$ we say that system (2.1) does not present resonance relations up to order s, inclusively.

The dependence of $\mathcal{H}_j$, $j \geq m$ with respect to θ occurs only through the s resonant angles generated by means of the vectors $k_1, \ldots, k_s$ with $0 \leq s \leq n-1$ where $\{k_1, \ldots, k_s\}$ is a basis of the $\mathbb{Z}$-module M_ω associated to H_2. Specifically,

$$M_\omega = k_1\,\mathbb{Z} + \cdots + k_s\,\mathbb{Z} = \left\{ j_1k_1 + \cdots + j_sk_s \,\middle|\, j_1, \ldots, j_s \in \mathbb{Z} \right\}.$$

It is clear that $M_\omega = \{0\}$ is equivalent to consider $\omega_1, \ldots, \omega_n$ linearly independent over $\mathbb{Q}$, that is, $M_\omega = \{0\}$ if and only if the system (3.1) does not possess resonances.

Notice that the set M_ω is finitely generated, then we can take a minimal set of generators, so $0 \leq s < n$ and the k_j are linearly independent.

At this point it is important to make precise the notions of single and multiple resonances.

Definition 1.21. *Assume that $M_\omega \neq \{0\}$. If M_ω is cyclic (equivalently $s = 1$) we say that the system (2.1) possesses single resonances and in the opposite case (or equivalently if $s > 1$) we say that the system possesses multiple resonances.*

The problem of knowing about the stability in the sense of Lyapunov of the null solution is in general an open problem. Only in particular cases we have methods in order to know the type of stability. Due to their importance, we will enunciate two general results:

- if the quadratic part H_2 is sign-definite in the autonomous case, the null solution is stable in the sense of Lyapunov (see, for example [21] or [70]).

- if there exists an eigenvalue of the linearized system with non zero real part, then the null solution is unstable (see, for example, [21] or [70]).

During this thesis, we will assume that the eigenvalues are pure imaginary, say $\pm\omega_1 i, \ldots, \pm\omega_n i$ (elliptic case). In the case H_2 is not sign-definite, the problem of characterizing the type of stability of the null solutions associated to (2.1) is too difficult, and in general, it is still an open problem. However, several partial results have been published with information about this question (most of them are published by Russian researchers). In fact, they give information about some subcases and analyze the normal form of the Hamiltonian system up to a finite order, but the type of stability provided, in general, is not in the Liapunov sense.

Other important concept is the stability of an equilibrium point in the Liapunov sense.

Definition 1.22. *It is said that the origin of* $\mathbb{R}^{2n}$ *in (2.1) is Liapunov stable if for every neighborhood* U *of the origin there is a neighborhood* U_1 *of the origin in* U *such that every solution* $x(t)$ *with* $x(0)$ *in* U_1 *is defined and in* U *for all* $t > 0$.

In order to enunciate our main results we need to recall the notion of Lie stability which was used in [26] applying normal form techniques. The idea was first introduced by Khazin in [42].

Definition 1.23. *We say that the origin of* $\mathbb{R}^{2n}$ *in (2.1) is Lie stable if there exists* $m > 2$ *such that the truncated Hamiltonian system in normal form associated to* $\mathcal{H}^j$ *is stable (in the sense of Liapunov) for any* $j \geq m$ *(arbitrary).*

Notice that both concepts, Birkhoff stability and Lie stability, are the same for an equilibrium with semisimple linear part, but Lie stability makes sense even in the non-diagonalizable case.

Regarding formal stability we provide the definition due to Moser [59].

Definition 1.24. *We say that the origin of* $\mathbb{R}^{2n}$ *in (2.1) is* formally stable *if there exists a real formal power series* $G(x)$ *(possibly divergent) which is an integral in the formal sense, and is positive definite near* $x = 0$.

In the cases of stability handled in [26, 28] it is proved that Lie stability implies formal stability. Here we prove that the same feature holds.

The null space of M_ω is a vector subspace of $\mathbb{R}^n$ spanned by the vectors $\{a_1, \ldots, a_d\}$ with $d = n - s$ that satisfy $a_i \cdot k_j = 0$ (see details in [26]). Setting $F_l = a_l \cdot I$, $l = 1, \ldots, d$, we get the independent (formal) first integrals of the normal form Hamiltonian (3.1). The set

$$S = \{I \,|\, F_1(I) = \ldots = F_d(I) = 0\}$$

is introduced for later use, noting that $0 \leq \dim S \leq s$. It was first given in [26] based in the geometric criterion for instability appearing in [34].

1.2.3 Stability results in two degrees of freedom

First, we consider the case when $n = 2$ and the frequencies ω_1, ω_2 of $\mathbb{H}$ have opposite sign i.e., the Hamiltonian has an indefinite quadratic part. Furthermore, we assume that the frequencies satisfy the resonance relation

$$m_1\omega_1 - m_2\omega_2 = 0$$

where m_1 and m_2 are relatively prime positive integers or $m_1 = m_2 = 1$ (i.e., resonance of order two). If $m_1 = m_2 = 1$, we assume also that the matrix of the linearized system is diagonalizable.
We write the Hamiltonian in action-angle variables $(I, \phi) := (I_1, I_2, \phi_1, \phi_2)$ defined as in (1.14) and assume that the Hamiltonian $\mathcal{H}$ is in normal form up to terms of order m where $m = 2l - 1$ or $m = 2l$, i.e.,

$$\mathcal{H}(I, \phi) = \mathbb{H}(I) + \mathcal{H}_4(I) + \cdots + \mathcal{H}_{2l-2}(I) + \mathcal{H}_m(I, m_1\phi_1 + m_2\phi_2) + \mathcal{H}^* \qquad (1.24)$$

where

- $\mathbb{H} = \omega_1 I_1 - \omega_2 I_2$
- $\mathcal{H}_{2j}$ is a homogeneous polynomial of degree j in I_1, I_2,
- $\mathcal{H}_m(I, m_1I_1 + m_2I_2)$ is a homogeneous polynomial of degree m in $\sqrt{I_1}$, $\sqrt{I_2}$ with coefficients which are finite Fourier series in the single angle $m_1\phi_1 + m_2\phi_2$,
- $\mathcal{H}^*$ denote terms of order greater than m in the variables $\sqrt{I_1}$, $\sqrt{I_2}$.
- $\mathcal{H}$ is a function analytic of the variables $\sqrt{I_1}$, $\sqrt{I_2}$.

Theorem 1.25 (Arnold's Theorem). *Let $D_{2j} = \mathcal{H}_{2j}(\omega_2, \omega_1)$. If for some $j = 2, \ldots, l-1$, $D_{2j} \neq 0$, then the equilibrium solution $q_i = p_i = 0$ is stable.*

When does not apply the Arnold's Theorem, i.e. $D_{2j} = 0$, for $j = 2, \ldots, l-1$ the term $\mathcal{H}_m$ will decide the stability or instability of the equilibrium. In this case, we introduce the function

$$\Psi(\phi) = \mathcal{H}_m(\omega_2, \omega_1, m_1\phi),$$

where $\phi = \phi_1 + \frac{m_2}{m_1}\phi_2$. Then we have the following result whose proof could be found in [13].

Theorem 1.26 (Markeev-Sokol'skii-Cabral-Meyer's Theorem). *If $\Psi(\phi) \neq 0$, for all ϕ, then the equilibrium solution $q_i = p_i = 0$ is stable. If Ψ has a simple zero, then the equilibrium solution is unstable.*

1.2.4 Stability results in n degrees of freedom

More recent works on Lie stable and unstable systems are due to dos Santos and coworkers [26, 28] where the authors establish several criteria dealing with Lie stable equilibria in cases of single and multiple resonance.

Let $k = (k_1, \ldots, k_n)$ be the vector of resonance. In the case $k_1, k_2, \ldots, k_n \geq 0$ or ≤ 0, without loss of generaliy, we will assume that $k_1 \neq 0$. We define the function

$$\begin{aligned} F_m(I_1, \phi) &= H^m\left(\tfrac{k_1}{k_1} I_1, \tfrac{k_2}{k_1} I_1, \ldots, \tfrac{k_n}{k_1} I_1, \phi\right) \\ &= A_4 I_1 + \cdots + A_{2l} I_1^l + \Psi_{|k|}(\phi) I_1^{|k|/2} + \Psi_{|k|+1}(\phi) I_1^{|k|/2+1/2} + \\ &\quad \cdots + \Psi_m(\phi) I_1^{m/2}, \end{aligned} \tag{1.25}$$

where $2l$ is an even natural number smaller or equal than $|k|$ and

$$A_{2j} = \frac{1}{k_1^j} H_{2j}(k), \; j = 2, \ldots, l, \quad \text{and} \quad \Psi_s(\phi) = \frac{1}{k_1^{s/2}} H_s(k, \phi), \tag{1.26}$$

where $s = |k|, \ldots, m$. The main result for single resonances where the linear part is diagonalizable is as follows.

Theorem 1.27. *Assume that the system* (3.1) *possesses a single resonance, with vector of resonance given by* $k = (k_1, \ldots, k_n)$.

- *(A) If there exist* $i, j \in \{1, \ldots, n\}$, $i \neq j$, *such that* $k_i k_j < 0$, *the null solution of* (3.1) *is Lie stable.*
- *(B) If* $k_1, k_2, \ldots, k_n \geq 0$ *or* ≤ 0 *and one of the coefficients* $A_{2j} \neq 0$ *for some* $j = 1, \ldots, |k| - 1$ *or in the case where* $A_{2j} = 0$ *for all* $j = 1, \ldots, |k| - 1$ *but the function in* (1.25) $\Psi_{|k|}(\phi) \neq 0$ *for all* ϕ, *then the null solution of* (3.1) *is Lie stable.*
- *(C) If* $k_1, k_2, \ldots, k_n \geq 0$ *or* ≤ 0 *and there exists* $\phi = \phi^*$ *such that* $\Psi_{|k|}(\phi^*) = 0$ *and* $\Psi'_{|k|}(\phi^*) \neq 0$ *(i.e.,* ϕ^* *is a simple zero), then the null solution of* (3.1) *is unstable in the Liapunov sense.*

We introduce the sets

$$S_1 = \{(I, \phi) \in \mathbb{R}^{n+1} : H^{|k|}(I, \phi) = 0\}$$

and

$$S_j = \{(I, \phi) \in \mathbb{R}^{n+1} : k_j I_1 - k_1 I_j = I_j = 0\}, \quad \text{with} \quad j = 2, \ldots, n.$$

Since $S_1 = \left(H^{|k|}\right)^{-1}(\{0\})$ and $\nabla_{(I,\phi)} H^{|k|}(I, \phi) \neq 0$, S_1 is an n-dimensional surface in $\mathbb{R}^{n+1}$. The sets $S_j (j = 2, \ldots, n)$ are n-dimensional planes in $\mathbb{R}^{n+1}$ or $S_j = \{I_1 = I_j = 0\}$. Note that $\{I_1 = \ldots = I_n = 0\} \subset S_1 \cap \ldots \cap S_n$. The geometric version of Theorem 1.27 is:

Theorem 1.28 (Geometric version). *Under the previous notation and considering that the system* (3.1) *possesses a single resonance, with vector of resonance given by* $k = (k_1, \dots, k_n)$*:*

(I) $S_1 \cap \dots \cap S_n = \{I_1 = \dots = I_n = 0\}$*, if and only if, there exist* $i, j \in \{1, \dots, n\}, i \neq j$*, such that* $k_i k_j < 0$ *or in the case* $k_1, k_2, \dots, k_n \geq 0$ *or* ≤ 0 *the function in* (1.25) *is such that* $\Psi_{|k|}(\phi) \neq 0$*.*

(II) There exists ϕ^* *such that* $(I, \phi^*) \in S_1 \cap \dots \cap S_n$ *(*$I \neq 0$*) and the interaction between* S_1 *and* $S_2 \cap \dots \cap S_n$ *at any point* (I, ϕ^*) *is transversal, if and only if,* $k_1, k_2, \dots, k_n \geq 0$ *or* ≤ 0 *and there exists* $\phi = \phi^*$ *a simple zero of* $\Phi_{|k|}(\phi)$*.*

Let $k^1, \dots, k^s$ to the vectors of resonance and $I_j = a_j \cdot I$ $(j = 1, \dots, n-s)$ with $a_j \cdot k^1 = \dots = a_j \cdot k^s = 0$

$$S = \{I; I_1(I) = \dots = I_{n-s}(I) = 0\},$$

and

$$S^m = \{I; H^m(I, \varphi) = 0, \forall \varphi\}.$$

We assume that $S \neq \{I = 0\}$. If $I \in S$, then $a_1 \cdot I = 0, \dots, a_{n-s} \cdot I = 0$ and since the vectors $a_1, \dots, a_{n-s}$ are linearly independent in $\mathbb{R}^n$, solving this previous system we can find $j_1, \dots, j_s \in \{1, \dots, n\}$ such that $I = I(I_{j_1}, \dots, I_{j_s})$ and $a_j \cdot I = 0$, $j = 1, \dots, n-s$. Now, we consider the function

$$F_m = H^m|_{S \times \mathbb{R}^s} = F_m(I_{j_1}, \dots, I_{j_s}, k^1 \cdot \varphi, \dots, k^s \cdot \varphi). \tag{1.27}$$

The main result for multiple resonances where the linear part is diagonalizable is the following.

Theorem 1.29 (Stability Theorem-Analytic Version). *With the previous notation, if* $S = \{I = 0\}$ *or* $S \neq \{I = 0\}$ *and if there exists* $m > 2$ *such that* $F_m(I_{j_1}, \dots, I_{j_s}, k^1 \cdot \varphi, \dots, k^s \cdot \varphi)$ *with* $I_{j_1}, \dots, I_{j_s} > 0$ *sufficiently small, then the null solution of* (3.1) *is Lie stable.*

We assume that for each $m > 2$ there exists φ^* such that

$$F_m(I_{j_1}, \dots, I_{j_s}, k^1 \cdot \varphi^*, \dots, k^s \cdot \varphi^*) = 0,$$

for all $I_{j_1}, \dots, I_{j_s} > 0$, sufficiently small. First, suppose that $|k^1| < |k^2| \leq \dots \leq |k^s|$ and $2|k^1| - 2 < |k^2|$. If $\eta = |k^1|$, then the truncated function $H^{2\eta-2}$ has the form

$$H^{2\eta-2} = H_2(I) + \dots + H_{2l}(I) + H_\eta(I, k^1 \cdot \varphi) + \dots + H_{2\eta-2}, (I, k^1 \cdot \varphi) \tag{1.28}$$

where $2l$ is an even natural number smaller than η. We suppose that H is in its Lie normal form up to order $2\eta-2$. In the case $k_1^1,\ldots,k_n^1 \geq 0$ with $k_1^1>0$, we consider the auxiliary function

$$\Psi(\phi)=\left(\frac{1}{k_1^1}\right)^{\eta/2} H_\eta(k^1,\phi), \tag{1.29}$$

where $\phi=k^1\cdot\varphi=k_1^1\varphi_1+\cdots+k_n^1\varphi_n$. Now we are in a position to enunciate the following result.

Theorem 1.30. *Under the previous notation, if $k_1^1,\ldots,k_n^1>0$, $H_4(k^1)=\ldots=H_{2l}(k^1)=0$ and if there exists ϕ^* such that $\Psi(\phi^*)=0$ and $\Psi'(\phi^*)\neq 0$, then the solution of* (3.1) *is unstable in the sense of Liapunov.*

The second case consists in assuming the existence of resonances such that $\eta=|k^1|=\ldots=|k^\mu|, 2\eta<|k^{\mu+1}|\leq\ldots\leq|k^s|$ with $s\geq\mu\geq 2$ and $k^1,\ldots,k^\mu$ do not have interactions. It is verified that the function $H^{2\eta}$ has the form

$$\begin{aligned} H^{2\eta} &= H_2(I)+\cdots+H_{2l}(I)+H_\eta(I,k^1\cdot\varphi,\ldots,k^\mu\cdot\varphi)+\\ &\cdots+H_{2\eta}(I,k^1\cdot\varphi,\ldots,k^s\cdot\varphi), \end{aligned} \tag{1.30}$$

where $2l$ is an even natural number smaller than η and there exist functions $H_i^j(I,k^j\cdot\varphi)$, and $H_i^0(I)$, $j=1,\ldots,\mu$, $i=2,\ldots,2\eta$, such that

$$H_i(I,k^1\cdot\varphi,\ldots,k^\mu\cdot\varphi)=H_i^0+H_i^1(I,k^1\cdot\varphi)+\cdots+H_i^\mu(I,k^\mu\cdot\varphi).$$

If $k_1^j,\ldots,k_n^j\geq 0$ and $k_1^j>0$, consider the auxiliary functions

$$\Psi_j(\phi_j)=\left(\frac{1}{k_1^j}\right)^{\eta/2} H_\eta(k^j,k^j\cdot\varphi), \quad j=1,\ldots,\mu, \tag{1.31}$$

where $\phi_j=j^j\cdot\varphi=k_1^j\varphi_1+\cdots+k_n^j\varphi_n$.

Theorem 1.31. *Under the above conditions, if there exists $j\in\{1,\ldots,\mu\}$ such that $k_1^j,\ldots,k_n^j\geq 0$, $k_1^j>0$, $H_4(k^j)=\ldots=H_{2l}(k^j)=0$ and if there exists ϕ_j^* such that $\Psi_j(\phi_j^*)=0$ and $\Psi_j'(\phi_j^*)\neq 0$ then the null solution of* (3.1) *is unstable in the sense of Liapunov.*

1.2.5 Chetaev's Theorem

Now we present some versions of Chetaev's theorem on a generalized cone, which are found in [44]. Consider the system of ordinary differential equations

$$\dot{u}=f(u),\ u\in\mathbb{R}^n,\ f(0)=0. \tag{1.32}$$

Theorem 1.32. *(Version of the Chetaev Theorem for Instability) Assume that there is a generalized n-dimensional cone K (see Figure 1.1) such that:*

1. *The trajectories may only enter on the lateral surfaces of K.*
2. *The equilibrium $u = 0$ is on the inferior basis of K and here the flows are invariant.*
3. *For the points of $Int(K)$ there exists a differentiable function of the type of the "Liapunov function" ($L(0) = 0, L(u) > 0$ for $u \neq 0$, $u \in Int(K)$) and $\dot{L}(u) > 0$ by virtue of the system (1.32) on $Int(K)$.*

Then the equilibrium $u = 0$ of (1.32) is unstable in the Liapunov sense.

The proof of this theorem can be found in [44].

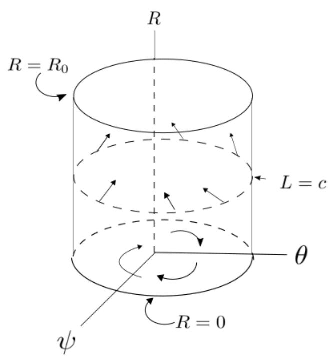


Figure 1.1: Representation of the cone K in Chetaev's Theorem 1.32.

Theorem 1.33. *Assume that the cone K with vertex in 0 (see Figure 1.2) has in cross-section the form of a cube (of dimension $n-1$) and that the following conditions are fulfilled:*

1. *The trajectories may only enter on each pair of opposite side faces of K.*
2. *For the points of $Int(K)$ there exists a differentiable function of the type of the "Liapunov function" ($L(0) = 0, L(u) > 0$ for $u \neq 0$, $u \in Int(K)$) and $\dot{L}(u) > 0$ by virtue of the system (1.32) on $Int(K)$.*

Then the equilibrium state $u = 0$ of (1.32) is unstable in the Liapunov sense.

The proof of this theorem can be found in [44].

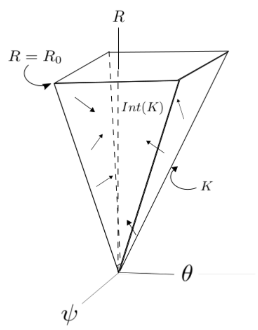


Figure 1.2: Representation of the cone K in Chetaev's Theorem 1.33.

Also, remember the converse of the theorem on asymptotically stability

Theorem 1.34. *If the equilibrium $u = 0$ of (1.32) is asymptotically stable, then there exists a positive definite and decreasing Liapunov function which is independent of t, with a negative definite derivative.*

The proof of this theorem can be found in [40].

1.3 Nekhoroshev theory

We are going to describe an important background for the study of asymptotic behavior or to get estimates of the solutions. It is the so called Nekhoroshev Theory [62].

1.3.1 Steep functions

In order to give the results on exponential time estimates of the Nekhoroshev Theory, we need to introduce the important concepts of steep, convex, quasi-convex, directionally quasi-convex, 3-jet nondegenerate and (γ, τ)-Diophantine steep function.

Definition 1.35. *Consider an open set Ω in $\mathbb{R}^n$; a real analytic function $f : \Omega \to \mathbb{R}$ is said to be steep at a point $I \in \Omega$ along an affine subspace Λ which contains I if there exist constants $C > 0, \delta > 0$ and $\alpha > 0$ such that along any continuous curva Γ in Λ connecting I to a point at a distance $r < \delta$ the norm of the projection of* *the gradient $\nabla f(x)$ onto the direction of Λ is greater than Cr^{α} at some point $F(t_*)$ with $||\Gamma(t) - I|| \leq r$ for all $t \in [0, t_*]$.*

The constants (C, δ) and α are respectively called the steepness coefficients and the steepness index. Under the previous assumptions, the functions f is said to steep at the point $I \in \Omega$ if, for every $m \in \{1, \ldots, n-1\}$, there exist positive constants C_m, δ_m and α_m such that f is steep at I along any affine subspace of dimension m containing I uniformly with respect to the coefficients (C_m, δ_m) and the index α_m.

Let H be an analytic Hamiltonian function with n degrees of freedom in a neighborhood of an elliptic equilibrium with constant frequency vector $\omega = (\omega_1, \ldots, \omega_n)$. Assuming that there are no resonances up to order N, we can write H as

$$H = H(I, \varphi) = k \circ I + f^{(N+1)}, \tag{1.33}$$

where $f^{(N+1)}$ is a convergent power series in (q, p) which begins with terms of degree $N+1$, $I = (I_1, \ldots, I_n)$ are the n action functions $I_j(q, p) = \frac{p_j^2 + q_j^2}{2}$, and

$$k(I) = k \circ I = k_2(I) + \sum_{j=2}^{[N]/2} k_{2j}(I), \quad k_2(I) = \omega \cdot I, \tag{1.34}$$

where $k(2j)$ is a homogeneous polynomial of degree j in $I_1, \ldots, I_n$, and $[\,]$ denotes the integer part. Thus, $k_4(I)$ is a quadratic form, which we will always write as

$$k_4(I) = \frac{1}{2} I \cdot A \cdot I, \tag{1.35}$$

where A being an $n \times n$-symmetric matrix. We remember the concepts of convex, quasi-convex, direcctionally quasi-convex and 3-jet non degenerate.

Definition 1.36. *The function $k(I)$ as in (1.34) is convex at $I = 0$, if the quadratic form $k_4(I)$ is either positive or negative definite.*

Definition 1.37. *The function $k(I)$ as in (1.34) is quasi-convex at $I = 0$, if the restriction of the quadratic form $k_4(I)$ to the plane orthogonal to ω is either positive or negative definite; equivalently, if*

$$\omega \cdot I = 0, \quad k_4(I) = 0 \quad \Rightarrow \quad I = 0. \tag{1.36}$$

Definition 1.38. *The function $k(I)$ as in (1.34) is directionally quasi-convex at $I = 0$ if the restriction of the quadratic form $k_4(I)$ to the plane orthogonal to ω does not vanish in the first octant:*

$$\omega \cdot I = 0, \quad k_4(I) = 0, \quad I_1, \ldots, I_n \geq 0 \quad \Rightarrow \quad I = 0. \tag{1.37}$$

Definition 1.39. *A function* $k(I) = \omega \cdot I + k_4(I) + k_6(I) + \cdots$ *as in* (1.34) *is said to be* 3*-jet nondegenerate at* $I = 0$ *if*

$$\omega \cdot I = 0, \quad k_4(I) = 0, \quad k_6(I) = 0 \quad \Rightarrow \quad I = 0. \tag{1.38}$$

For $m \in \{1, \dots, n\}$, we denote by $Graff_R(n, m)$ the m-dimensional affine Grassmannian over $\overline{B}_R^{(n)} \subset \mathbb{R}^{(n)}$of radius $R > 0$ around the origin) and

$$Graff_R^K(n, m) \subset Graff_R(n, m)$$

is the set of rational subspaces of dimension m in $\mathbb{R}^n$ whose direction is spanned by integer vectors of length $||\vec{k}||_1 = |k_1| + \cdots + |k_n| \leq K$ for a given $K \in \mathbb{N}^*$.

Definition 1.40. *A differentiable function* f *defined on a neighbourhood of* $\overline{B}_R^{(n)} \subset \mathbb{R}^n$ *is said to be* (γ, τ)*-Diophantine steep with two positive constants* γ *and* τ *if, for any* $m \in \{1, \dots, n\}$*, there exist an index* $\alpha_m \geq 1$ *and coefficients* $C_m > 0, \delta > 0$ *such that along any affine subspace* $\Lambda \in Graff_R^K(n, m)$ *and any continuous curve* Γ *from* $[0, 1]$ *to* $\Lambda_m \cap B_R$ *with* $||\Gamma(0) - \Gamma(1)|| = r \leq \delta_m(\gamma/K^\tau)$*, we have that there exists* t_* *such that*

$$||\Gamma(0) - \Gamma(t)|| \leq r \quad \textit{for all} \quad t \in [0, t_*],$$
$$||proj_{\vec{\Lambda}_m}(\nabla f(\gamma(t_*)))|| \geq C_m r^{\alpha_m},$$

where $\vec{\Lambda}_m$ *is the direction of* Λ_m*.*

We also remember the Diophantine condition on the vector $\sigma = (\sigma_1, \dots, \sigma_d)$, that is, it is supposed that there are fixed constants $c > 0$ and $\nu > d - 1$ such that

$$\forall k \in \mathbb{Z}^d \setminus \{0\}, |k \cdot \sigma| \geq c|k|^{-\nu}. \tag{1.39}$$

1.3.2 Results of exponential time estimates

We present the main results on exponential time estimates without condition Diofanthine for functions: steep, convex, directionally quasi-convex and 3-jet non-degenerate and with condition Diofanthine. Let

$$H(I, \varphi, \varepsilon) = H_0(I) + \varepsilon H_1(I, \varphi, \varepsilon), \tag{1.40}$$

be a perturbed Hamiltonian 2π-periodic in φ and H_0 an integrable Hamiltonian. In his celebrated article [62], Nekhoroshev proved the following result.

Theorem 1.41 (Nekhoroshev, [62]). *If the unperturbed Hamiltonian $H_0(I)$ is a steep function, then there exist a, b, c such that in the perturbed Hamiltonian system for a sufficiently small perturbation we have*

$$|I(t) - I(0)| < \varepsilon^b, \quad for \quad 0 \leq t \leq \frac{1}{\varepsilon} exp\left(\varepsilon^a / c\right). \tag{1.41}$$

Here a, b, c are positive constants depending on the characteristics of the unperturbed Hamiltonian.

More precisely,

$$\begin{aligned} & a = \tfrac{2}{12\zeta + 3n + 14}, \quad b = \tfrac{3a}{2\alpha_{n-1}} \\ & \zeta = \left[\alpha_1 \left(\alpha_2 \left(\ldots \left(\alpha_{n-3}(n\alpha_{n-2} + n - 2) + n - 3\right) + \cdots\right) + 2\right) + 1\right] - 1. \end{aligned} \tag{1.42}$$

The result on exponential time estimates for convex functions is as follows.

Theorem 1.42 (Benettin, Fassò, Guzzo, [7]). *Assume that the Hamiltonian (1.34) is convergent for $|w|^\infty, |z|^\infty \leq R_C$, with some R_C, and that the quadratic form $I \cdot AI$ is convex. Then, there exist positive constants $R \leq R_C, \varepsilon_*, \varepsilon_0, c$ and C such that any motion of the system (1.34), with real initial data $(w(0), z(0)) = (w(0), -i\overline{w}(0))$ with*

$$\varepsilon = \frac{|I(0)|^\infty}{R^2} \leq \varepsilon_0,$$

satisfies

$$|I(t)|^\infty \leq c\left(\frac{\varepsilon}{\varepsilon_*}\right)^{\frac{1}{n}} R^2, \quad for \quad |t| \leq C \exp\left[\frac{1}{2}\left(\frac{\varepsilon_*}{\varepsilon}\right)^{\frac{1}{n}}\right]$$

as well as

$$|I(t)|^\infty \leq c\left(\frac{\varepsilon}{\varepsilon_*}\right)^{\frac{1}{2}+\frac{1}{2n}} R^2, \quad for \quad |t| \leq C \exp\left[\frac{1}{2}\left(\frac{\varepsilon_*}{\varepsilon}\right)^{\frac{1}{2n}}\right].$$

Here $|I|^\infty = \max_{1 \leq j \leq n} |I_j|$.

The Nekhoroshev estimates for directionally quasi-convex and 3-jet nondegenerate functions are given in [8].

Theorem 1.43 (Benettin, Fassò, Guzzo, [8]). *Assume that $N \geq 4$ and that $k(I)$ is directionally quasi-convex. Then, for ε sufficiently small, any motion with initial conditions such that $|I(0)| \leq \varepsilon$ satisfies estimates (1.41) with $a = b = 1/n$ as well as with $a = 1/(2n), b = 1/2$.*

Theorem 1.44 (Benettin, Fassò, Guzzo, [8]). *Consider the Hamiltonian* (1.33) *for $n = 3$, with $k(I)$ as in* (1.34) *and $N \geq 8$. Assume that k is* 3*-jet nondegenerate. Then, estimates* (1.41) *hold for any ε small enough, with $a = \min\left(\frac{N-7}{20}, \frac{N+1}{36}\right)$ and $b = 1$.*

The result on exponential time estimates for (γ, τ)-Diophantine steep functions is given in [64].

Theorem 1.45 (Niederman, [64]). *Let $\mathcal{H}(I, \varphi) = h(I) + \varepsilon f(I, \varphi)$ be a nearly integrable Hamiltonian analytic on the complex neighbourhood $V_{r,s}\mathcal{P} \subset \mathcal{C}^{2n}$ with an integrable par $h(I)$ which is (γ, τ)-Diophantine steep. Consider*

$$\beta = \frac{1}{2(1 + n^n \alpha_1 \ldots \alpha_{n-1})}, \quad a = \frac{\beta}{1+\tau}, \quad b = \frac{\beta}{\alpha_n}.$$

There exists a positive constant C which depends on n, M, R, s and τ but not on ε and γ such that for a small enough perturbation $\varepsilon \leq C Inf(\gamma^{1/a}, \gamma^{1/b})$ and for any orbit of the perturbed system with initial conditions $(I(t_0, \varphi(t_0))) \in B_R \times \mathbb{T}^n$ far enough from the boundary of B_R, we have

$$||I(t) - I(t_0)|| \leq (n+1)^2 \varepsilon^b \quad |t| \leq \exp\left(\frac{s}{6}\varepsilon^{-a}\right).$$

The Nekhoroshev estimates for quasi-convex and τ, γ-Diophantine functions are given in [23].

Theorem 1.46 (Delshams, Gutiérrez, [23]). *Let $H(\phi, I) = \omega \cdot I + f(\phi, I)$ real analytic on $\mathcal{D}_\rho(\mathcal{G})$, and assume that the vector ω is τ, γ-Diophantine for some $\tau \geq n - 1$ and $\gamma > 0$. Assume*

$$\varepsilon := ||f||_{\mathcal{G},\rho} \leq \varepsilon_0 := \frac{\gamma \rho_2}{244}.$$

Then, for every trajectory $(\phi(t), I(t))$ of H, with $(\phi(0), I(0)) \in \mathbb{T}^n \times \mathcal{G}$, one has

$$|I(t) - I(0)| \leq \frac{\rho_2}{r\rho_1}\left(\frac{\varepsilon}{\varepsilon_0}\right)^{1/(\tau+1)} \quad \textit{for} \quad |t| \leq \frac{2}{\gamma}\left(\frac{\rho_1}{24}\left(\frac{\varepsilon_0}{\varepsilon}\right)^{1/(\tau+1)}\right).$$

Theorem 1.47 (Delshams, Gutiérrez, [23]). *Let $H(\phi, I) = h(I) + f(\phi, I)$ real analytic on $\mathcal{D}_\rho(\mathcal{G})$, let $\omega = grad\ h$, and assume that*

$$\left|\frac{\partial^2 h}{\partial I^2}\right|_{\mathcal{G},\rho_2} \leq M, \quad |\omega|_{\mathcal{G}} \leq L.$$

Assume also that h is m-quasiconvex on $\mathcal{U}_{\rho_2}(\mathcal{G})$. Let $\gamma > 0$ given, and assume:

$$\lambda \leq \frac{23 M^2 \rho_2}{m}, \quad \varepsilon := ||f||_{\mathcal{G},\rho} \leq \varepsilon_0 := \frac{m^{4n-1}\widehat{\rho}\lambda^2}{2^{24n-2M^{4n}}},$$

where we write $\widehat{\rho} := \min(\rho_1, 2/\sqrt{n})$*. Then, for every trajectory* $(\phi(t), I(t))$ *of* H*, with* $(\phi(0), I(0)) \in \mathbb{T}^n \times \mathcal{G}$ *and satisfying* $|\omega(I(0))| \geq \lambda$*, one has*

$$|I(t) - I(0)| \leq \rho_2 \left(\frac{\varepsilon}{\varepsilon_0}\right)^{1/2n}, \quad \mathit{for} \quad |t| \leq \frac{4}{L} \exp\left(\frac{m\rho_1}{24M}\left(\frac{\varepsilon_0}{\varepsilon}\right)^{1/2n}\right).$$

Recently, in [32] it is considered a real analytic Hamiltonian of the form

$$\mathcal{H}(x) = \mathbb{H}(x) + \mathcal{K}(x). \tag{1.43}$$

It is possible to arrange $\mathbb{H}$ so that it has the form

$$\mathbb{H} = \xi_1 F_1 + \cdots + \xi_{n-s} F_{n-s}, \tag{1.44}$$

where the ξ_l are linear combinations of the μ_i with the condition that $\xi = (\xi_1, \ldots, \xi_d)$ with $d = n-s$, is a nonresonant frequency vector, feature that is always guaranteed by the construction of the F_i from the $\mathcal{M}_\mu$.

Proposition 1.48. *Given the Hamiltonian system associated to* (1.43) *with* $\mathbb{H}$ *as in* (1.44)*, the following statements are equivalent:*

(i) *The set* $S = \{J = 0\}$*.*

(ii) *There is a linear combination of the* d *formal integrals* F_l *for the normal form Hamiltonian* $\overline{\mathcal{H}}$ *related to* $\mathcal{H}$ *such that is a positive definite quadratic form in* x*.*

(iii) *The Hamiltonian* $\mathbb{H}$ *can be written as*

$$\mathbb{H} = \sigma_1 Q_1 + \cdots + \sigma_d Q_d, \tag{1.45}$$

where all the Q_l *are nonnull formal integrals of* $\overline{H}$ *and positive semidefinite quadratic forms in* x*. The frequency vector* $\sigma = (\sigma_1, \ldots, \sigma_d)$ *is nonresonant.*

Theorem 1.49. *If the real analytic Hamiltonian* (1.43) *has* $\mathbb{H}$ *in the form* (1.44) *satisfying conditions (i), (ii) or (iii) of Proposition 1.48, while the frequency vector* σ *satisfies the Diophantine condition* (1.39)*, then there exist* $C > 0, K > 0, a > 1$ *and* $\rho_0 > 0$ *such that for all* $\rho \in (0, \rho_0)$*, and for all* x_0 *with* $|x_0| < \rho$ *we have*

$$|x(t, x_0)| < a\rho \quad \mathit{for\ all} \quad 0 \leq t \leq T = C\rho \exp\left(\frac{K}{\rho^{1/(2(\nu+1))}}\right).$$

This result is important, because the estimates depend only on the linearized system and not on the higher order terms as in KAM theory and it is not required any steepness or convexity conditions as in Nekhoroshev theory.

<table>
<tr><th>Author</th><th>Year</th><th>Function</th><th>a</th><th>b</th></tr>
<tr><td>Nekhoroshev, [62]</td><td>1977</td><td>Steepness condition</td><td>$\frac{2}{12\zeta+3n+14}$</td><td>$\frac{3a}{2\alpha_{n-1}}$</td></tr>
<tr><td rowspan="2">Lochak, Neishtadt, [49]</td><td rowspan="2">1992</td><td rowspan="2">quasi convex</td><td colspan="2">$\frac{1}{2n}$</td></tr>
<tr><td colspan="2">$\frac{1}{2}$</td></tr>
<tr><td rowspan="3">Poschel, [65]</td><td rowspan="3">1993</td><td rowspan="3">quasi convex</td><td colspan="2">$\frac{1}{2n}$</td></tr>
<tr><td>μa</td><td>$\mu a+\frac{1-\mu}{2}$</td></tr>
<tr><td>$\frac{1}{2n}$</td><td>$\frac{1}{2}$</td></tr>
<tr><td rowspan="2">Delshams, Gutiérrez, [23]</td><td rowspan="2">1996</td><td>τ,γ-Diophantine</td><td colspan="2">$\frac{1}{\tau+1}$</td></tr>
<tr><td>quasi-convex</td><td colspan="2">$\frac{1}{2n}$</td></tr>
<tr><td rowspan="5">Benettin, Fassò, Guzzo, [8], [7]</td><td rowspan="5">1998</td><td rowspan="2">convex</td><td colspan="2">$\frac{1}{n}$</td></tr>
<tr><td>$\frac{1}{2n}$</td><td>$\frac{1}{2}+\frac{1}{2n}$</td></tr>
<tr><td rowspan="2">directionally quasi convex</td><td colspan="2">$\frac{1}{n}$</td></tr>
<tr><td>$\frac{1}{2n}$</td><td>$\frac{1}{2}$</td></tr>
<tr><td>3-jet nondegenerate</td><td>$\min\left(\frac{N-7}{20},\frac{N+1}{36}\right)$</td><td>$1$</td></tr>
<tr><td rowspan="2">Niederman, [64]</td><td rowspan="2">2007</td><td>steep</td><td colspan="2">$\frac{1}{(2n-1)\alpha_1\ldots\alpha_{n-1}+1}$</td></tr>
<tr><td>γ,τ-Diophantine steep</td><td>$\frac{1}{2(1+\tau)(1+n^n\alpha_1\ldots\alpha_{n-1})}$</td><td>$\frac{1}{2\alpha_n(1+n^n\alpha_1\ldots\alpha_{n-1})}$</td></tr>
<tr><td>Bounemoura, Marco, [10]</td><td>2011</td><td>quasi convex</td><td>$\frac{1}{2(n-1)}+\delta$</td><td>$\delta(n-1)$</td></tr>
<tr><td>Bounemoura, Niederman, [11]</td><td>2012</td><td>Simoultaneous Diophantine Morse</td><td colspan="2">$\frac{1}{3(2(n+1)\tau)^n}$</td></tr>
<tr><td>Benettin, Chierchia, Guzzo, [37]</td><td>2016</td><td>steep</td><td>$\frac{1}{2\alpha_1\ldots\alpha_{n-2}}$</td><td>$\frac{1}{\alpha_{n-1}}$</td></tr>
</table>

Table 1.1: Summary of some results on exponential time estimates.

1.3.3 Algorithm for detecting directional quasi-convexity

In general, the verification of the conditions quasi-convexity, directionally quasi-convexity and 3-jet nondegenerate is not an easy problem. In [30] and [39] the authors describe an algorithm for the numerical verification of steepness, a necessary property for the application of Nekhoroshev's theorem, of functions of three degrees of freedom. Next, in order to make this work self contained, we describe briefly the algorithm in [39].

The algorithm we are going to construct allows us verify the steepness of the function:

$$h(I) = \sum_{i=1}^{n} \omega_i I_i + \frac{1}{2} \sum_{i,j=1}^{n} A_{ij} I_i I_j + \frac{1}{6} \sum_{i,j,k=1}^{n} B_{ijk} I_i I_j I_k + \frac{1}{24} \sum_{i,j,k,l=1}^{n} C_{ijkl} I_i I_j I_k I_l, \quad (1.46)$$

for $n = 3, 4$, with ω_i, A_{ij}, B_{ijk}, C_{ijkl} known coefficients, in a neighborhood of $\bar{I} = 0$. We assume $\omega = (\omega_1, ..., \omega_n) \neq 0$ and denote by Λ the linear space orthogonal to ω. This algorithm represents an extension of the algorithm provided by Benettin, Fasso and Guzzo in [8], testing the 3-jet nondegeneracy and (following the terminology introduced in [8]) directional quasi-convexity of a three degree of freedom Hamiltonian

$$h(I) = \sum_{i=1}^{n} \omega_i I_i + \frac{1}{2} \sum_{i,j=1}^{n} A_{ij} I_i I_j + \frac{1}{6} \sum_{i,j,k=1}^{n} B_{ijk} I_i I_j I_k.$$

All steps described below may be explicitly implemented with any suitable computer algebra system. The first three steps constitute the algorithm constructed in [8]. The last one represents the extension of the algorithm to the case of a function h which is 3-jet degenerate at the origin.

1) We perform a rotation of the variables I in order to carry the vector ω into the first coordinate axis, and denote by R the rotation matrix. Then we take the appropriate 2×2 submatrix A_Λ of RAR^T, which represents the restriction of the Hessian matrix A to the space Λ. If the two eigenvalues of A_Λ are both positive or negative, then we conclude that $h(I)$ is quasi-convex at the origin.

2) We suppose $h(I)$ is not quasi-convex at the origin. We compute the vectors $v \in \Lambda\{0\}$ such that $h^2[v, v] = 0$ as follows. Let $\lambda_1 \leq \lambda_2$ be the eigenvalues of A_Λ, and let $x = (x_1, x_2)$, $y = (y_1, y_2)$ be their eigenvectors (of unitary norm). Since any vector $v \in \Lambda\{0\}$ has the representation $v = R^T(0, d_1, d_2)$, with some $d = (d_1, d_2) \in \mathbb{R}^2 \setminus \{0\}$, the condition $h^2[v, v] = 0$ is satisfied if $d \in \mathbb{R}^2 \setminus \{0\}$ solves

$$A_\Lambda d \cdot d = 0. \quad (1.47)$$

Since A_Λ is symmetric, we can diagonalize it by an orthogonal matrix S : $A_\Lambda = SDS^T$, where D is diagonal with $D_{i,i} = \lambda_i$, and (1.47) becomes:

$$\lambda_1\omega_1^2 + \lambda_2\omega_2^2 = 0, \tag{1.48}$$

where $\omega = (\omega_1, \omega_2) \in \mathbb{R}^2 \setminus \{0\}$ is such that $\omega = S^T d$. We distinguish between different cases, depending on the values of λ_1, λ_2.

A) $\lambda_1 < 0 < \lambda_2$: (1.48) determines two lines through the origin, which, in original coordinates, are generated by the unit vectors:

$$\begin{aligned} v^A &= R^T(0, d^A) \quad \textit{with} \quad d^A = \sqrt{\tfrac{\lambda_1}{\lambda_1-\lambda_2}}y + \sqrt{\tfrac{-\lambda_2}{\lambda_1-\lambda_2}}x, \\ v^B &= R^T(0, d^B) \quad \textit{with} \quad d^B = \sqrt{\tfrac{\lambda_1}{\lambda_1-\lambda_2}}y - \sqrt{\tfrac{-\lambda_2}{\lambda_1-\lambda_2}}x, \end{aligned}$$

If it happens that: $h^3[v^A, v^A, v^A]$, $h^3[v^B, v^B, v^B] \neq 0$, then $h(I)$ is 3−jet nondegenerate at the origin. If, instead, it happens that $h^3[v^A, v^A, v^A] = 0$ or $h^3[v^B, v^B, v^B] = 0$, then $h(I)$ is 3−jet degenerate.

B) One of the two eigenvalues λ_1, λ_2 vanishes: we first suppose $\lambda_1 = 0$, so that $\lambda_2 > 0$ and (1.48) is solved by the vectors $\omega = (\omega_1, 0)$ with $\omega_1 \in \mathbb{R}$, and in particular by $\omega = (1, 0)$. Consequently, the vector $v \in \Lambda \setminus \{0\}$ defined by $v = R^T(0, x)$ satisfies $h^2[v, v] = 0$, and therefore $h(I)$ is 3−jet nondegenerate at the origin if and only if $h^3[v, v, v] \neq 0$. Similarly, if $\lambda_2 = 0$, $h(I)$ is 3−jet nondegenerate at the origin if and only if $h^3[v, v, v] \neq 0$, with $v = R^T(0, y)$.

C) $\lambda_1 = \lambda_2 = 0$: in this case (1.48) is solved by all the vectors $\omega \in \mathbb{R}^2 \setminus \{0\}$. Therefore, the unit vectors $v \in \Lambda \setminus \{0\}$ satisfying $h^2[v, v] = 0$ are

$$v^\gamma = R^T(0, d^\gamma) \quad \text{with} \quad d^\gamma = x\cos\gamma + y\sin\gamma,$$

for all $\gamma \in [0, 2\pi)$. As a consequence, $h(I)$ is 3−jet nondegenerate at the origin if and only if for each $\gamma \in [0, 2\pi)$, the vector v^γ satisfies $h^2[v^\gamma, v^\gamma, v^\gamma] \neq 0$.

3) From step (2), we obtained all the vectors $v \in \Lambda \setminus \{0\}$ satisfying $h^2[v, v] = 0$. We can therefore directly check, if needed, if $h(I)$ is directionally quasi-convex at the origin.

Chapter 2

Formal stability of elliptic equilibria in Hamiltonian systems with exponential time estimates

We deal with elliptic equilibria in Hamiltonian systems with n degrees of freedom, establishing a criterion to determine their formal stability and providing asymptotic estimates on the solutions starting nearby.

The method consists in calculating a linear subspace of $\mathbb{R}^{2n}$ that we call S and that is contained into the orthogonal space related to the frequency vector. Next the normal form Hamiltonian is computed up to a suitable order and we check whether the truncated Hamiltonian vanishes only at the origin in the linear subspace. If this occurs we obtain a type of formal stability that is called Lie stability.

To our knowledge there are no examples of systems that are formally stable but not Liapunov stable, which gives an idea of the strength of formal stability in the setting of nonlinear stability of equilibria.

Formal stability of elliptic equilibria was started by Siegel [69] and Moser [58, 59, 60] who established conditions on the quadratic terms of the Hamiltonians to achieve formal stability. Glimm [36] proved formal stability provided the quartic terms in normal form and in action-angle variables do not depend on the angles and are indeed a definite function in terms of the actions. Bryuno [12] refined previous results getting a criterion for formal stability of Hamiltonians based on the quadratic and quartic terms.

Other members of the Russian school also contributed significantly to the research in formal stability starting in the decade of the 70. We quote the pioneering work by Khazin [42, 43] who established the concept of Lie stability, although he named it Birkhoff stability. More papers dealing with formal stability and instability for several cases managing resonant situations are [51, 73, 45, 46].

Based on Nekhoroshev theory [62] for steep functions in the setting of the stability of elliptic equilibria, several authors [35, 22, 48] established results on bounds for exponentially long times on the actual solutions near an equilibrium of an analytic Hamiltonian system. These bounds were improved later in [7, 8, 63, 66]. Recently the theory of stability has been enlarged in [68, 39] to deal with some degenerate situations where steepness is obtained from higher-order terms, and thus Nekhoroshev estimates apply. As well the papers [33, 9] deal respectively with very sharp estimates in the case of Diophantine conditions among the frequencies and the relationship between the nonlinear stability of elliptic equilibria and the existence of KAM tori nearby.

In addition to the above, in a series of papers Guzzo and coworkers, Niederman and Bounemoura have relaxed the hypotheses to get Nekhoroshev estimates, allowing the part of the Hamiltonian depending only on the actions to be non-steep. More precisely, Guzzo *et al.* [38] introduced the notion of rational convexity, which roughly means that the convexity property is tested only on the subspaces of fast drift. This idea has been generalised by Niederman [64] under the name of Diophantine steepness condition (see also an equivalent concept in [11]), which is a weak condition of transversality that involves only the affine subspaces spanned by integer vectors. This property leads to exponential estimates of stability of Nekhoroshev type. Checking these conditions on a specific problem is not usually an easy task.

Surprisingly there is almost no connection in the literature between Nekhoroshev stability of elliptic equilibria and formal stability. Indeed, excepting the pioneering papers by Glimm and Bryuno, on the one hand the works related to formally stable systems do not consider the issue of getting estimates that measure the validity of the nonlinearly stable solutions. On the other hand the studies on elliptic equilibria from the point of view of Nekhoroshev theory have obtained very sharp bounds on the solutions but they do not deal with the existence of positive definite first integrals.

More recent papers on Lie stable and unstable systems are due to dos Santos and coworkers [26, 28, 29] where the authors establish several criteria dealing with Lie stable equilibria in cases of resonances. They also treat instability using suitable Chetaev functions [21]. Another related work treating a particular case of Lie stability is [54]. The instability analysis using the invariant ray technique is developed in [42, 43, 73, 15, 16]. Asymptotic estimates for Lie stable systems where the corresponding linear subspace S is trivial are carried out in [32].

In the present work we aim at getting Lie stability with the weakest possible assumptions. This is achieved by exploiting the algebraic structure of the linear part of the equation as much as we can. We do not need to check whether the truncated normal form Hamiltonian vanishes for all non-null vectors of the orthogonal space related to the frequency vector, but only for the subspace S. Thus the lower the dimension of S is, the more cases of Lie stable systems we get. This allows us to obtain Lie stable systems for which exponential time estimates apply but such that they do not satisfy the conditions of Nekhoroshev estimates appearing in [38, 64, 11], as we show in Section 2.4. We do not tackle the instability issue although we intend to enlarge the existing theorems in the framework of our approach.

The estimates we obtain are based on a recent paper by Chartier *et al.* [20] where the authors determined error bounds for adiabatic invariants of Hamiltonian systems. More specifically the variation of these invariants by a truncation may remain bounded over exponentially long time intervals. Therefore, our point of view is independent of the approach of other methods mainly based in Nekhoroshev theory.

2.1 Statement of the main results

Consider the autonomous Hamiltonian system with n degrees of freedom

$$\dot{x} = \mathcal{J}\nabla H(x), \tag{2.1}$$

such that the origin of the phase space is an equilibrium solution, $\mathcal{J}$ is the standard $2n \times 2n$ symplectic matrix of Hamiltonian theory [53] and $H = H(x)$ is a real analytic function of $x = (x_1, \ldots, x_n, y_1, \ldots, y_n)$. It is assumed that the Taylor series of H in a neighborhood of the origin is

$$H = H_2 + H_3 + \cdots + H_j + \cdots\,, \tag{2.2}$$

where H_j represents a homogeneous polynomial of degree j in x, that is,

$$H_j = \sum_{|k|+|l|=j} h_{kl}\; x^k y^l, \tag{2.3}$$

with $k = (k_1, \ldots, k_n) \in \mathbb{Z}^n$, $l = (l_1, \ldots, l_n) \in \mathbb{Z}^n$, $|k| = |k_1| + \cdots + |k_n|$, $|l| = |l_1| + \cdots + |l_n|$, $h_{kl} = h_{k_1 \cdots k_n l_1 \cdots l_n}$, $x^k = x_1^{k_1} \cdots x_n^{k_n}$ and $y^l = y_1^{l_1} \cdots y_n^{l_n}$.

We state our main result on stability for the elliptic equilibria. In the following $|\;|$ stands for the Euclidean norm.

Theorem 2.1. *(A) Suppose there is an integer $j \geq 3$ with $\mathcal{H}^j(I, \phi_1, \ldots, \phi_s) \neq 0$ for all $I \in S \setminus \{0\}$, $\{\phi_1, \ldots, \phi_s\} \in \mathbb{T}^s$ and there is not an index i with $3 \leq i < j$ such that $\mathcal{H}^i(I, \phi_1, \ldots, \phi_s)$ changes sign for some $I \in S \setminus \{0\}$, $\{\phi_1, \ldots, \phi_s\} \in \mathbb{T}^s$, where $|I|$ is small enough. Then the origin of $\mathbb{R}^{2n}$ is Lie stable for the Hamiltonian system (2.1).*

(B) Suppose there is an integer $i \geq 3$ such that $\mathcal{H}^i(I, \phi_1, \ldots, \phi_s)$ changes sign for some $I \in S \setminus \{0\}$, $\{\phi_1, \ldots, \phi_s\} \in \mathbb{T}^s$, where $|I|$ is small enough. Then there is not an index $j > i$ such that $\mathcal{H}^j(I, \phi_1, \ldots, \phi_s) \neq 0$ for $I \in S \setminus \{0\}$, $\{\phi_1, \ldots, \phi_s\} \in \mathbb{T}^s$ with $|I|$ sufficiently small.

The estimates we obtain are based on a recent paper by Chartier *et al.* [20] where the authors determined error bounds for adiabatic invariants of Hamiltonian systems. More specifically the variation of these invariants by a truncation may remain bounded over exponentially long time intervals. Therefore, our point of view is independent of the approach of other papers, mainly based in Nekhoroshev theory.

The quadratic part in terms of the formal first integrals F_k assumes the form

$$H_2(I) = \sum_{k=1}^{d} \sigma_k F_k(I), \tag{2.4}$$

where the σ_k are linear combinations of the ω_j.

With the aim of getting the time estimates we need to impose a *Diophantine condition* on the vector $\sigma = (\sigma_1, \ldots, \sigma_d)$; that is, we suppose that there are fixed constants $c > 0$ and $\nu > d - 1$ such that

$$\forall k \in \mathbb{Z}^d \setminus \{0\}, \quad |k \cdot \sigma| \geq c|k|^{-\nu}. \tag{2.5}$$

We state our result on the exponential time estimates for the elliptic equilibria when formal stability holds from Theorem 2.1.

Theorem 2.2. *If the real analytic Hamiltonian (2.2) has the origin of $\mathbb{R}^{2n}$ as a formally stable equilibrium according to hypotheses (A) of Theorem 2.1, while the frequency vector σ satisfies the Diophantine condition (2.5), then there exist $C > 0$, $K > 0$, $a > 1$ and $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$, and for all x_0 with $|x_0| < \varepsilon$ we have*

$$|x(t, x_0)| < a\, \varepsilon^{2/j} \quad \textit{for all } t \textit{ with } \quad 0 \leq t \leq T = C \exp\left(\frac{K}{\varepsilon^{1/(\nu+1)}}\right).$$

2.2 Proof of the stability

Proof of Theorem 2.1. (A) Define V as

$$V = F_1^2 + \cdots + F_d^2 + (\mathcal{H}^p)^2. \tag{2.6}$$

This function is a first integral of the Hamiltonian system associated to $\mathcal{H}^p$ for every $p \geq 2$. We have that $V = 0$ if and only if $F_1(I) = \ldots = F_d(I) = 0$ and $\mathcal{H}^p = 0$. Thus it is enough to take $I \in S$.

When $S = \{0\}$ we get Lie stability straightforwardly since $\mathcal{H}^p$ evaluated at $I \in S$ is trivially zero and the last term of (2.6) can be dropped.

When $S \neq \{0\}$ we perform a stretching of coordinates, say $x \to \varepsilon y$ with $\varepsilon > 0$ small, that in action-angle variables reads as $I \to \varepsilon^2 J$, $\theta \to \theta$. (Notice that $|I|$ small in (A) is equivalent to consider ε small.) To make the change symplectic we multiply (3.1) by ε^{-2} arriving at

$$H(J,\theta,\varepsilon) = H_2(J) + \cdots + \varepsilon^{2l-4}\mathcal{H}_{2l-2}(J) + \varepsilon^{m-2}\mathcal{H}_m(J,\theta) + \cdots . \qquad (2.7)$$

Assuming the hypotheses in (A) hold, for $\varepsilon > 0$ sufficiently small then $\mathcal{H}^p = 0$ if and only if $J = 0$ since there is an integer j with $3 \leq j < p$ such that $\mathcal{H}^j \neq 0$ for all $J \in S \setminus \{0\}$, thus the addition of higher-order terms $\mathcal{H}_{j+1}$, $\mathcal{H}_{j+2}$, ..., cannot change the sign of $\mathcal{H}^j$. By Liapunov Theorem [47] the null solution is stable for the Hamiltonian system associated to $\mathcal{H}^p$. Considering $\mathcal{H}^q$ with $q \geq p$ and taking V as before but changing $\mathcal{H}^p$ by $\mathcal{H}^q$ it follows that V is a first integral of $\mathcal{H}^q$ which is positive definite. Since q is arbitrary, the null solution of (2.1) is Lie stable.

(B) When there is an index $i \geq 3$ with $\mathcal{H}^i(I,\phi_1,\ldots,\phi_s) = 0$ for $I \in S \setminus \{0\}$, $|I|$ small and $\{\phi_1,\ldots,\phi_s\} \in \mathbb{T}^s$, then $\mathcal{H}^i$ changes sign and from (2.7) it is clear that the higher-order terms of the normal form cannot alter this feature provided ε is taken small enough. Thus, one cannot find $\mathcal{H}^j$ with $j > i$ such that (A) can be applied. □

Remark 2.3. *If in the function V introduced in (2.6), $\mathcal{H}^p$ is replaced by the normal form up to infinity $H_2 + \mathcal{H}_3 + \cdots + \mathcal{H}_p + \cdots$ (i.e. formally), then V becomes a formal first integral of Hamiltonian (1.22) which is positive definite, thus Hamiltonian system (2.1) is formally stable.*

Remark 2.4. *When determining the sign of $\mathcal{H}^j$ for $I \in S \setminus \{0\}$ if there is an index $i < j$ such that $\mathcal{H}^i = 0$ for certain I^*, ϕ_i^*, $i = 1,\ldots,s$ then we have to evaluate $\mathcal{H}^{i+1}$ only at I^*, ϕ_i^* and proceed in this way order by order until reaching order j. For an illustration see the last example of Section 2.4.*

Remark 2.5. *We can refine the hypotheses of the Theorem by considering s_j an integer in $[0,s]$ such that $\mathcal{H}^j$ depends only on s_j angles (without loss of generality their first s_j angles) thus we write $\mathcal{H}^j(I,\phi_1,\ldots,\phi_{s_j})$.*

2.3 Asymptotic estimates

There exist only a few results dealing with estimates on formally stable equilibria. As classical achievements we report the papers by Moser [60] and Glimm [36], and recently the paper [32] that accounts for the exponentially large estimates on time for Lie stable systems for which $S = \{0\}$.

For the cases of Lie stable equilibria provided in Theorem 2.1 we give time estimates of exponential type, similar to those of Nekhoroshev theory. Our result is based upon the time estimates for adiabatic invariants established by Chartier *et al.* in [20].

Noticing that for $k = 1, \dots, d$, $F_k(I) = \varepsilon^2 F_k(J)$, Hamiltonian (2.7) has d formal first integrals given by $F_k(J)$. Moreover, one can construct other first integrals whose main part is $F_k(J)$. Our goal is to provide estimates on the time evolution of these first integrals. To achieve this we introduce a few ingredients before stating the result of Chartier *et al.* on adiabatic invariants in a form suited to our needs.

The normal form Hamiltonian (2.7) is rewritten in terms of the rectangular coordinates y as

$$H(y, \varepsilon) = H_2(y) + \cdots + \varepsilon^{2l-4}\mathcal{H}_{2l-2}(y) + \varepsilon^{m-2}\mathcal{H}_m(y) + \cdots , \tag{2.8}$$

here the Hamiltonian $H_2(y)$ in (2.8) can be expressed in terms of the F_k by means of

$$H_2(y) = \sum_{k=1}^{d} \sigma_k F_k(y), \tag{2.9}$$

as in (2.4). At this point we introduce some more notation. Let $\mathcal{N} = B_R$ be the open ball of radius $R > 0$ centered on 0 in $\mathbb{R}^{2n}$. Given a solution $y = y(t, y_0, \varepsilon)$ of the system related to (2.8) with initial condition y_0 in $\mathcal{N}$, let $\gamma = \gamma(y_0, \varepsilon)$ be the solution's first time of escape from $\mathcal{N}$, i.e.

$$\gamma = \inf\{t > 0 \,\big|\, |y(t, y_0, \varepsilon)| \geq R\}. \tag{2.10}$$

Given $\gamma > 0$ and $T > 0$, we set

$$D = [0, \gamma) \cap [0, T], \tag{2.11}$$

thus, D is the shortest of the two intervals.

Let

$$I_i^p(y, \varepsilon) = F_i(y) + \sum_{k=3}^{p} \varepsilon^{k-2} I_{i,k}(y), \quad i = 1, \dots, d,$$

be first integrals of Hamiltonian (2.8) truncated at order p, where $I_{i,k}(y)$ are homogeneous polynomials in y of degree k. Then the following result appeared as Corollary 3.6 in [20].

Theorem 2.6 (Chartier, Murua and Sanz-Serna)**.** *Let the real analytic system associated to (2.8) satisfy the Diophantine condition (2.5) and let $y_0 \in \mathcal{N}$. Then* *there are constants $C > 0$ and $K > 0$ such that for small enough $\varepsilon > 0$, there is a positive integer p such that for arbitrary $\kappa > 0$ and for $i = 1, \ldots, d$,*

$$|I_i^p(y(t, y_0, \varepsilon), \varepsilon) - I_i^p(y_0, \varepsilon)| < \kappa^2 \quad \textit{for all} \quad t \in D = [0, \gamma) \cap [0, T],$$

where

$$T = C\,\kappa^2 \exp\left(\frac{K}{\varepsilon^{1/(\nu+1)}}\right).$$

The integer p refers to the order to which the normal form has to be carried out in order to get the required estimate on the time; it is the N of Theorem 3.5 and Corollary 3.6 in [20]. The parameter κ is independent of ε and is not necessarily small as it can be deduced from the proof of Theorem 3.5 and from Corollary 3.6. Finally it is assumed that $\varepsilon \in (0, \varepsilon_1)$, where $\varepsilon_1 > 0$ is an appropriate threshold.

Theorem 2.6 was established in the context of the averaging procedure devised by the authors in [18, 19] to deal with vector fields (both dissipative and Hamiltonian) from the point of view of the design and analysis of numerical integrators. It is stressed that the averaging (or normal form transformation) accomplished in [20], under the assumptions of the above theorem the corresponding remainder is exponentially small. In [32] a time estimate based on Chartier *et al.* was established for the case $S = \{0\}$, while here we enlarge this result for the elliptic equilibria that are formally stable using the criterion of Theorem 2.1.

We prove now Theorem 2.2.

Proof of Theorem 2.2. We prove the bounds on the system derived from the Hamiltonian in normal form (3.1) and this will imply the bounds on the system (2.1), as the passage to normal form involves only a finite number of steps.

We suppose that there is an integer $j \geq 3$ such that $\mathcal{H}^j$ does not vanish for $I \in S \setminus \{0\}$. Let $p \geq j$ be the integer to which the normal form Hamiltonian (1.22) has been obtained. First we prove that given a small enough $\tilde{\varepsilon}_0 > 0$ there are positive constants α, β, γ such that whenever $|x| \leq \tilde{\varepsilon}_0$ we have

$$\alpha|x|^{2j} \leq V(x), \quad (\mathcal{H}^p(x))^2 \leq \beta|x|^4, \quad (F_l(x))^2 \leq \gamma|x|^4, \;\; l = 1, \ldots, d, \tag{2.12}$$

where V is defined in (2.6).

For $|x| \leq \tilde{\varepsilon}_0 < 1$ one obtains $(\mathcal{H}^p(x))^2 = (H_2(x))^2 + \mathcal{O}(|x|^5)$, then by selecting $\beta > \max\{\omega_1^2, \ldots, \omega_n^2\}/4$ we ensure that $\beta|x|^4 \geq (\mathcal{H}^p(x))^2$ for $\tilde{\varepsilon}_0$ sufficiently small.

It is straightforward to notice that for $l = 1, \ldots, d$, one has $|F_l(x)| \leq \sqrt{\gamma_l}|x|^2$ for some $\gamma_l > 0$, thus γ is chosen as $\max\{\gamma_1, \ldots, \gamma_d\}$.

When $x = 0$ the first inequality of (2.12) holds trivially, then we consider $x \neq 0$. Setting $W(x) = V(x) - \alpha|x|^{2j}$ we get $W(x) = \sum_{l=1}^{d}(F_l(x))^2 + (H_2(x))^2 + \mathcal{O}(|x|^5)$.

When x is small and in correspondence with the action $I \notin S$ it is clear that $W(x) \geq 0$ as the terms of order 4 in x do not vanish. Hence we consider x such that its corresponding $I \in S \setminus \{0\}$, ending up with

$$\begin{aligned} W(x) &= (\mathcal{H}^p(x))^2 - \alpha|x|^{2j} = (\mathcal{H}^j(x))^2 - \alpha|x|^{2j} + \mathcal{O}(|x|^{2j+1}) \\ &= (\mathcal{H}^j(x) - \sqrt{\alpha}|x|^j)(\mathcal{H}^j(x) + \sqrt{\alpha}|x|^j) + \mathcal{O}(|x|^{2j+1}). \end{aligned}$$

Since hypotheses A of Theorem 2.1 hold, without loss of generality we assume $\mathcal{H}^j(x) > 0$ for x related with $I \in S \setminus \{0\}$, thus it is enough to prove that $U(x) = \mathcal{H}^j(x) - \sqrt{\alpha}|x|^j$ is positive for x small enough and an adequate choice of α. Applying the stretching $x \to \varepsilon y$ to U we get

$$U^*(y,\varepsilon) = \frac{1}{\varepsilon^3}\left(\varepsilon^2\mathcal{H}^j(y,\varepsilon) - \varepsilon^j\sqrt{\alpha}|y|^j\right) = \mathcal{H}_3(y)+\varepsilon\mathcal{H}_4(y)+\cdots+\varepsilon^{j-3}\left(\mathcal{H}_j(y) - \sqrt{\alpha}|y|^j\right), \tag{2.13}$$

where $U^*(y,\varepsilon) = \varepsilon^{-3}U(\varepsilon y)$, $\mathcal{H}^j(y,\varepsilon) = H_2(y) + \varepsilon\mathcal{H}_3(y) + \cdots + \varepsilon^{j-2}\mathcal{H}_j(y)$ and we have taken into account that $\mathcal{H}^j(x) = \varepsilon^2\mathcal{H}^j(y,\varepsilon)$ and $H_2(y) = 0$ for y associated to $J \in S$ with $J = \varepsilon^{-2}I$. According to the part of (B) in Theorem 2.1, we notice that each $\mathcal{H}_k(y) \geq 0$ for all k in (2.13) and moreover $\mathcal{H}_j(y) > 0$ for y related to $J \in S \setminus \{0\}$, otherwise we would not achieve Lie stability. The positiveness of $\mathcal{H}_j(y)$, a homogeneous polynomial in y of degree j, ensures the existence of $\alpha > 0$ such that $\mathcal{H}_j(y) \geq \sqrt{\alpha}|y|^j$. Therefore $U^*(y,\varepsilon) \geq 0$ and $V(x) \geq \alpha|x|^{2j}$ where $|x| \leq \tilde{\varepsilon}_0$ with $\tilde{\varepsilon}_0$ small enough.

From (2.12) we conclude that for $|x| \leq \tilde{\varepsilon}_0$ one gets

$$\alpha|x|^{2j} \leq V(x) \leq (\beta + d\gamma)|x|^4 \tag{2.14}$$

where α is chosen smaller than $\beta + d\gamma$.

The second step consists in proving that when $t \in D$ we get

$$|\mathcal{H}^p(x(t))| < Q\varepsilon^2 \quad \text{and} \quad |F_l(x(t))| < Q'\varepsilon^2, \tag{2.15}$$

with certain positive constants Q, Q' independent of ε that will be specified later. From the identities $\mathcal{H}^p(x) = \varepsilon^2\mathcal{H}^p(y,\varepsilon)$, $F_l(x) = \varepsilon^2 F_l(y)$ we deduce

$$\mathcal{H}^p(y,\varepsilon) = H_2(y) + \sum_{k=3}^{p}\varepsilon^{k-2}\mathcal{H}_k(y) = \sum_{l=1}^{d}\sigma_l F_l(y) + \sum_{k=3}^{p}\varepsilon^{k-2}\mathcal{H}_k(y).$$

Now $\mathcal{H}^p$ is written down as follows:

$$\mathcal{H}^p(y,\varepsilon) = \sigma_1 F_1^*(y,\varepsilon) + \sum_{l=2}^{d}\sigma_l F_l(y)$$

with

$$F_1^*(y,\varepsilon) = F_1(y) + \frac{1}{\sigma_1}\sum_{k=3}^{p} \varepsilon^{k-2}\mathcal{H}_k(y).$$

The fact that $\mathcal{H}^p$ and F_1 are formal integrals of H in (2.8) implies that $\sum_{k=3}^{p} \varepsilon^{k-2}\mathcal{H}_k(y)$ is also a formal integral of the same Hamiltonian, hence F_1^* is a formal first integral as well.

Applying the estimate given in Theorem 2.6 to F_1^*, F_2, ..., F_d, it is readily deduced that for $t \in D$ and for arbitrary $\kappa > 0$:

$$\begin{aligned} |\mathcal{H}^p(y(t),\varepsilon) - \mathcal{H}^p(y_0,\varepsilon)| &\leq |\sigma_1|\,|F_1^*(y(t),\varepsilon) - F_1^*(y_0,\varepsilon)| + \sum_{l=2}^{d} |\sigma_l||F_l(y(t)) - F_l(y_0)| \\ &< \kappa^2 \sum_{l=1}^{d} |\sigma_l| = E, \end{aligned}$$

where $y_0 = \varepsilon^{-1}x_0$. Thus

$$|\mathcal{H}^p(x(t)) - \mathcal{H}^p(x_0)| < E\varepsilon^2.$$

Using the second inequality in (2.12) we find that

$$|\mathcal{H}^p(x(t))| < |\mathcal{H}^p(x_0)| + E\varepsilon^2 \leq \sqrt{\beta}|x_0|^2 + E\varepsilon^2 < Q\varepsilon^2 \quad \text{with} \quad Q = E + \sqrt{\beta},$$

when $t \in D$.

As $|F_l(y(t)) - F_l(y_0)| < \kappa^2$ one has $|F_l(x(t)) - F_l(x_0)| < \kappa^2\varepsilon^2$, $l = 1, \dots, d$, therefore $|F_l(x(t))| < |F_l(x_0)| + \kappa^2\varepsilon^2$, but we know that $|F_l(x_0)| \leq \sqrt{\gamma}|x_0|^2 < \sqrt{\gamma}\varepsilon^2$ when t is in D, hence $|F_l(x(t))| < Q'\varepsilon^2$ with $Q' = \kappa^2 + \sqrt{\gamma}$.

Using the previous inequalities we arrive at

$$\alpha|x(t)|^{2j} \leq V(x(t)) = \sum_{l=1}^{d}(F_l(x(t)))^2 + (\mathcal{H}^p(x(t)))^2 < dQ'^2\varepsilon^4 + Q^2\varepsilon^4 = Q''\varepsilon^4 \quad (2.16)$$

for $t \in D$, and then

$$|x(t)| < a\,\varepsilon^{2/j} \quad \text{where} \quad a = \left(\frac{Q''}{\alpha}\right)^{1/(2j)}, \quad (2.17)$$

stressing that $a > 1$ because $\alpha < \beta + d\gamma < Q''$. It is remarked that the inequalities in (2.16) apply when $|x(t)| \leq \tilde{\varepsilon}_0$, thus setting $a\,\varepsilon^{2/j} < \tilde{\varepsilon}_0$ we get the bound $\varepsilon < (\tilde{\varepsilon}_0/a)^{j/2}$ and choose $\varepsilon_0 = \min\{(\tilde{\varepsilon}_0/a)^{j/2}, \varepsilon_1\}$, where $\varepsilon_1 > 0$ is the threshold guaranteed by Theorem 2.6.

As $\kappa > 0$ is arbitrary we set it equal to one converting T of Theorem 2.6 into

$$T = C \exp\left(\frac{K}{\varepsilon^{1/(\nu+1)}}\right).$$

Finally we want to show that $\gamma > T$ and thus $D = [0, T]$. Assume the contrary, that is, take $\gamma \leq T$ so that $D = [0, \gamma)$ and consider $\varepsilon < \min\{\varepsilon_0, (R/(2a))^{j/(2-j)}\}$. Now by assumption, $|y(t, y_0, \varepsilon)| \nearrow R$ as $t \nearrow \gamma$. Applying Theorem 2.6 and estimate (2.17) given above, we arrive at $|y(t, y_0, \varepsilon)| < a\,\varepsilon^{2/j-1} < R/2$ for all $t \in [0, \gamma)$, which is a contradiction. It follows that $\gamma > T$, so $D = [0, T]$ as desired. $\square$

Remark 3.1. Theorem 2.2 generalises Theorem 5.1 of [32], because when $S = \{0\}$, in the proof of Theorem 2.2 we can set $j = 2$, $\kappa = 1$ and drop $\mathcal{H}^p$, getting the estimates of Theorem 5.1 in [32].

Remark 3.2. The estimate (2.17) gets worse for the solution $x(t)$ as j grows, indicating the fact that the more terms one needs to conclude Lie stability the worse the bounds on the solutions are. However the exponential estimates on the time T do not depend on j.

Remark 3.3. As stated in [20], when $d = 1$ it is possible to set $\nu = 0$ in (2.5) because small divisors cannot arise and the Diophantine condition is dropped. In this case the time T can be very large, moreover the constants C and K are better. In particular this is the situation of fully resonant Hamiltonians.

Remark 3.4. When $j = 4$ and $\mathcal{H}^4$ depends only on the actions I, our estimates are not directly comparable with those of Nekhoroshev type. This would be an interesting point of research.

Remark 3.5. In case that Lie stability is not accomplished (or even instability is obtained) one can still deduce asymptotic bounds for some action coordinates as follows. Since we have d first integrals F_l satisfying Theorem 2.6, there always exists a linear change of coordinates from I, θ to $\tilde{I}, \tilde{\theta}$ such that $\tilde{I}_l = F_l$ for $l = 1, \ldots, d$. Therefore these actions satisfy Chartier *et al.*'s estimates and from the proof of Theorem 2.2 one gets bounds of the form $|\tilde{I}_l(x(t, x_0)) - \tilde{I}_l(x_0)| < \varepsilon^2$ for exponentially large time.

Remark 3.6. It would be desirable to lessen the Diophantine hypothesis stated above, replacing it by another non-resonant condition, but currently the Diophanticity of the vector σ is required.

2.4 Implications and examples

2.4.1 The case $n = 2$

For two degrees of freedom, our result is the same as the stability part in Cabral-Meyer's Theorem (Theorem 4.1 of [13]) that includes Arnold's Theorem as well as other results of Alfriend [2, 3, 4] and Markeev [**?**, 51]. Therefore Lie stability becomes Liapunov stability. The reason of this is that the function $\mathcal{H}^j$ of Theorem 2.1 agrees with the function Ψ in Theorem 4.1 of [13].

2.4.2 The case $S = \{0\}$

In this situation one always obtains Lie stability, see details in [28]. More specifically, part (A) of Theorem 2.1 applies trivially and one considers $V = \sum_{k=1}^{d} F_k^2$. A particular situation is that of definite H_2, where Liapunov stability holds by applying Dirichlet Theorem [25]. All the first integrals F_k are written as linear combinations of the form $\sum_j \alpha_{j,k} I_j$ and without loss of generality we assume $\alpha_{j,k} > 0$. Hence for every $k = 1, \ldots, d$, $F_k = 0$ if and only if $I_j = 0$ for all j appearing in F_k, therefore $S = \{0\}$. Similarly in the absence of resonances among the ω_i, since $\omega_1, \ldots, \omega_n$ are linearly independent over $\mathbb{Q}$, the first integrals are $F_j = I_j$ with $j = 1, \ldots, n$, then $d = n$ and the set S is null. Normal stability introduced in [54] is also a particular case of Lie stability where $S = \{0\}$. The estimates obtained in [32] when $S = \{0\}$ are comparable to those provided by Theorem 2.2, see Remark 3.1.

For example, the Hamiltonian function

$$H = (1 - \sqrt{2})I_1 - \sqrt{2}I_2 + (2 - \sqrt{2})I_3 - \sqrt{2}I_4 + \cdots ,$$

where ... refers to higher-order terms in normal form starting at order three, has resonance vectors $k_1 = (2, 0, -1, -1)$, $k_2 = (0, 2, 0, -2)$ and two formal integrals, namely, $F_1 = I_1 + 2I_2 + 2I_4$ and $F_2 = I_1 + I_2 + I_3 + I_4$. Hence it is easily deduced that S is null, concluding Lie stability. As $H_2 = \sigma_1 F_1 + \sigma_2 F_2$ with $(\sigma_1, \sigma_2) = (-1, 2 - \sqrt{2})$, which is a Diophantine vector, according to Remark 3.1 the estimates of Theorem 2.2 apply with $j = 2$.

We stress that when $S = \{0\}$ Lie stability can be achieved even for Hamiltonians whose first nonlinear term $\mathcal{H}_3$ is non-zero. If this occurs then $\mathcal{H}_3$ in terms of action-angle coordinates depends on angles because d'Alembert character [53] is satisfied.

2.4.3 Lie stability decided from terms than depend on actions

Our theory extends previous results in the sense that we can get Lie stability for Hamiltonian systems that even do not satisfy the conditions needed in Nekhoroshev theory, obtaining Lie stable systems under rather weak conditions.

More specifically we assume that hypotheses of (A) hold for some $j > 3$ and such that $\mathcal{H}^j$ does not depend on angles, so $j < m$ in (3.1). When $j = 4$ and $\mathcal{H}^4(I) = H_2(I) + \mathcal{H}_4(I)$ is directionally quasi-convex, that is, H_2 and H_4 never vanish simultaneously for $I \neq 0$, then exponential time estimates apply [8]. Choosing $I \in S \setminus \{0\}$ then $H_2(I) = 0$ and if directional quasi-convexity holds, $\mathcal{H}_4(I) \neq 0$, so $\mathcal{H}^4(I) \neq 0$ implying that Theorem 2.1 applies. Thus, directional quasi-convex of elliptic equilibria is a particular case of Lie stability. This argument can be extended to include 3-jet non-degenerate functions [68, 39], that are a particular situation of steep functions, and Nekhoroshev theory remains valid. Considering $I \in S \setminus \{0\}$ with $\mathcal{H}^6(I)$ a 3-jet non-degenerate function, one arrives at the condition $\mathcal{H}^6(I) \neq 0$, hence Lie stability is satisfied. Unfortunately, according to Schirinzi and Guzzo [68], higher-order jets need extra hypotheses in order to guarantee steepness and although the corresponding assumptions have been established only for 4-jets while steepness from orders higher than 4 are hard to analyse.

As said in the introduction, exponential estimates of Nekhoroshev type have been obtained recently by several authors relaxing steepness conditions, see the papers [38, 64, 11]. The hypotheses that one has to check are not straightforward, but they essentially involve to check whether some Hessian matrices obtained in suitable affine subspaces of $\mathbf{R}^{2n}$ are non-degenerate. Translated to the setting of elliptic equilibria it implies a significative restriction in the terms $\mathcal{H}_4(I)$, $\mathcal{H}_6(I)$, ..., $\mathcal{H}_j(I)$. However we can handle examples of Lie stable systems with exponential bounds that are very degenerate, as we shall see later.

The example below illustrates that we can have Lie stability without satisfying steepness condition. Consider the Hamiltonian with three degrees of freedom

$$H = H_2 + \mathcal{H}_4 + \cdots , \tag{2.18}$$

with

$$\begin{aligned} H_2 &= -\tfrac{1}{10}(6+\sqrt{6})I_1 + \tfrac{1}{10}(-2+3\sqrt{6})I_2 + I_3, \\ \mathcal{H}_4 &= I_1^2 + \alpha I_2^2 + I_3^2 + I_1 I_2 + I_1 I_3 + I_2 I_3, \end{aligned}$$

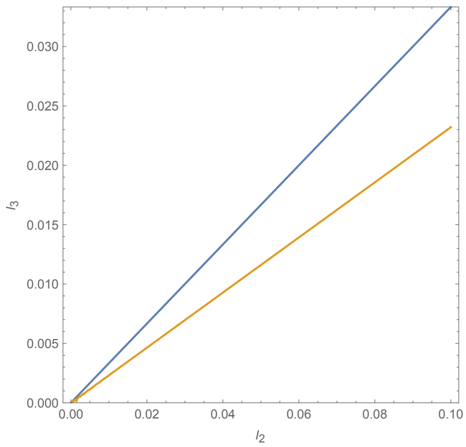
with α a real parameter. Hamiltonian H is supposed to be in normal form up to a certain order. The $\mathbb{Z}$-module M_ω is spanned by $k_1 = (3, 1, 2)$ and the functions $F_1 = -2I_1 + 3I_3$, $F_2 = -I_1 + 3I_2$ are the formal first integrals of the system associated to H, so $d = 2$. The corresponding set S related to the quadratic terms of (2.18) is given by $\{(3I_3, I_3, 2I_3) \mid I_3 \geq 0\}$, thus $\dim S = 1$. As $\mathcal{H}^4(I) = H_2(I) + \mathcal{H}_4(I) = (24+\alpha)I_3^2$ for $I \in S$, one gets $\mathcal{H}^4 \neq 0$ when $I_3 \neq 0$, $\alpha \neq -24$. Thence, applying Theorem 2.1 the origin of $\mathbb{R}^6$ is Lie stable for the Hamiltonian system associated to (2.18), provided $\alpha \neq -24$. However, the Hamiltonian function (2.18) is convex only if $\alpha > 1/3$. This condition can be somewhat relaxed, assuming directional quasi-convexity as introduced in [8]. After some straightforward computations we conclude that $\mathcal{H}_4$ is directionally quasi-convex for $\alpha > 2(\sqrt{6}-4)/3$, which is the bound for α in order to get Nekhoroshev type of stability for steep systems. Nevertheless, the previous bound can be extended applying the notion of rational convexity of Guzzo *et al.* guzzo2006diffusion, where convexity has to be tested only in the affine planes of fast drift, which are subspaces of integer vectors of dimensions up to $n-1$. In this case, it is enough to take into account the one-dimensional subspace spanned by k_1. By proceeding as in [38] one arrives at exponential stability when $\alpha \neq -24$, i.e. the same restriction we found to achieve Lie stability. Finally, in case Lie stability holds we notice that $H_2 = \sigma_1 F_1 + \sigma_2 F_2$ with $\sigma_1 = 1/3$, $\sigma_2 = (3\sqrt{6}-2)/30$, thus (σ_1, σ_2) is Diophantine and the estimates of Theorem 2.2 hold.

As the following example we take

$$H = H_2 + \mathcal{H}_{10} + \cdots = 3I_1 - 2I_2 + 6I_3 - I_2^5 + \cdots . \tag{2.19}$$

In this case the resonance vectors are $k_1 = (2, 0, -1)$, $k_2 = (0, 3, 1)$ and $S = \{(2(I_2 - 3I_3), 3I_2, 3I_3) \mid I_2 \geq 3I_3 \geq 0\}$, the only formal first integral is $F_1 = 3I_1 - 2I_2 + 6I_3$, hence $d = 1$. Considering $I \in S$ it is clear that $\mathcal{H}^{10}(I) = H_2(I) + \mathcal{H}_{10}(I) = -I_2^5 = 0$ if and only if $I_2 = 0$, but then $I_1 = I_3 = 0$, thus $\mathcal{H}^{10} < 0$ for $I \in S \setminus \{0\}$ and Lie stability holds. However the system is too degenerate to obtain stability from Nekhoroshev theory. In fact as $\mathcal{H}_4 = 0$ steepness condition fails and the more relaxed conditions of rational convexity [38] and Diophantine steepness [64] fail as well since we can select a two-dimensional affine subspace of $\mathbb{R}^6$ so that the corresponding Hessian matrix is degenerate. Regarding the estimates on $x(t)$ and the time T they can be straightforwardly applied as $d = 1$, thus no Diophantine condition is required in this case, see Remark 3.3.

Let us consider $H = H_2 + \mathcal{H}_6 + \cdots$ where H_2 is as in the previous case and $\mathcal{H}_6 = 40I_3^3 - I_2^3/2$. Then $\mathcal{H}_6(I^*) = 0$ for points of the form $I^* = (I_1^*, I_2^*, I_3^*) = ((2/3 - 10^{-1/3})I_2^*, I_2^*, I_2^*/(2 \cdot 10^{1/3}))$, which are in the interior of S for $I_2^* > 0$. Thus we cannot build a positive definite first integral V as defined in (2.6) and then, applying part (B) of Theorem 2.1, Lie stability for the origin of $\mathbb{R}^6$ cannot be achieved by adding higher-order terms to $\mathcal{H}^6$. In Fig. 2.1 we compare the effect of taking two different $\mathcal{H}_6$ with the same H_2, leading to different behaviours. Theorem 2.6 can be applied to F_1 getting exponential time estimates on the time T, see Remark 3.5.



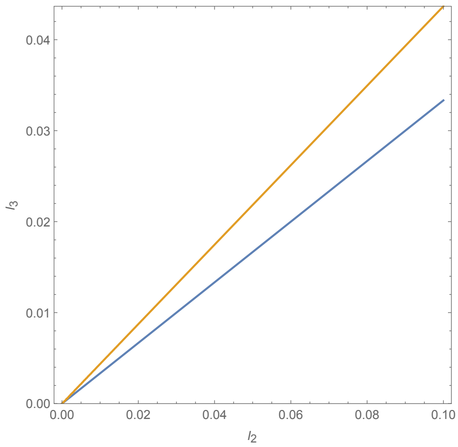


Figure 2.1: On the left we plot the curves $I_2 = 3I_3$ (blue) and $40I_3^3 = I_2^3/2$ (orange) showing that $\mathcal{H}_6$ changes sign in S, hence Lie stability cannot be achieved. On the right we consider $\mathcal{H}_6 = 4I_3^3 - I_2^3/3$ and plot the curves $I_2 = 3I_3$ (blue) and $4I_3^3 = I_2^3/3$ (orange) showing that the origin of $\mathbb{R}^6$ is Lie stable for the Hamiltonian $H_2 + \mathcal{H}_6 + \dots$.

For the next example we choose a Hamiltonian (1.22) with n degrees of freedom and such that $\mathcal{H}_3 = 0$ and $\mathcal{H}_4$ independent of the angles and a quasi-convex function of I, so Nekhoroshev stability of the origin of $\mathbb{R}^{2n}$ holds. Furthermore we suppose that there is only one resonant angle, thus $s = 1$ and one can take the integer vector $k_1 = (k_{11}, \dots, k_{1n})$ as the resonance vector. In this case $d = n - 1$ and the corresponding formal first integrals F_j are obtained as $F_j = k_{11}I_j - k_{1j}I_1$ with $j = 2, \dots, n$ where without loss of generality we take $k_{11} \neq 0$. If $I \in S$, we get $I_j = (k_{1j}/k_{11})I_1$ for $j = 2, \dots, n$. Thus, Hamiltonian (1.22) evaluated at $I \in S$ assumes the form

$$H = \frac{1}{k_{11}^2}\mathcal{H}_4(k_{11}, k_{12}, \dots, k_{1n})I_1^2 + \cdots . \tag{2.20}$$

Since $\mathcal{H}_4$ is quasi-convex, $\mathcal{H}^4 \neq 0$ provided $I_1 \neq 0$ and the null solution is Lie stable for the Hamiltonian system associated to H. This should be expected as Nekhoroshev stability of elliptic equilibria implies Lie stability. Regarding the estimates issue one can apply the estimates provided in [7, 63, 66].

2.4.4 Lie stability decided from terms than depend on angles

As the Hamiltonian $\mathcal{H}^j$ of Theorem 2.1 can depend on ϕ_k, our result generalises Theorem 3.1 of [26] and Theorem 1.1 of [28], dealing with the situations of single and multiple resonance, respectively. Furthermore, our result remains valid for any $\mathcal{H}^j$ satisfying the hypotheses in (A) regardless of the dependence or not of some intermediate Hamiltonians $\mathcal{H}_i$ with $i \leq j$ with respect to some angles ϕ_k.

An example of a Hamiltonian system with three degrees of freedom that has multiple resonances of orders 4 and 5 is derived from the Hamiltonian function

$$H = H_2 + \mathcal{H}_4 + \mathcal{H}_5 + \cdots , \tag{2.21}$$

where

$$\begin{aligned} H_2 &= \left(\sqrt{2} - 7\right) I_1 + 3\left(7 - \sqrt{2}\right) I_2 + \tfrac{1}{2}\left(5\sqrt{2} - 35\right) I_3, \\ \mathcal{H}_4 &= I_1^{3/2}\sqrt{I_2}\cos\phi_1 + I_1^2 + I_2^2 + I_3^2 + I_1 I_2 + I_1 I_3 + I_2 I_3, \\ \mathcal{H}_5 &= \sqrt{I_1} I_2 I_3 \cos\phi_2, \end{aligned}$$

and $\phi_1 = 3\theta_1 + \theta_2$, $\phi_2 = \theta_1 + 2\theta_2 + 2\theta_3$. In this case, the resonance vectors are $k_1 = (3, 1, 0)$, $k_2 = (1, 2, 2)$; $F_1 = 2I_1 - 6I_2 + 5I_3$ is the corresponding formal first integral and the set S is given by $\{(6I_2 - 5I_3, 2I_2, 2I_3) \mid 0 \leq I_3 \leq 6I_2/5\}$ with $\dim S = 2$. Taking $I \in S$ we get

$$\mathcal{H}_4(I, \phi_1) = 52I_2^2 + 19I_3^2 - 54I_2I_3 + \sqrt{2I_2}(6I_2 - 5I_3)^{3/2}\cos\phi_1.$$

At this point we notice that when $I_3 \in [0, 6I_2/5]$ then $52I_2^2 + 19I_3^2 - 54I_2I_3 > |\sqrt{2I_2}(6I_2 - 5I_3)^{3/2}|$ from where it is readily deduced that $\mathcal{H}^4(I, \phi_1)$ is positive for $I \in S \setminus \{0\}$ and any $\phi_1 \in \mathbb{T}$. Thus, the origin of $\mathbb{R}^6$ is Lie stable for the Hamiltonian system associated to (2.21). We remark that for $\mathcal{H}_5$ we could have chosen any Hamiltonian in normal form in terms of I and ϕ_2 provided it satisfies the d'Alembert character. As in this case we only have a first integral, F_1, the estimates of Theorem 2.2 on $x(t)$ and T apply with $j = 4$. As $d = 1$ no Diophantine condition is needed for getting the estimates.

Our next example represents a Hamiltonian with multiple resonances of orders 3 and 4 for which the Hamiltonian function is

$$H = H_2 + \mathcal{H}_3 + \mathcal{H}_4 + \cdots , \tag{2.22}$$

where

$$\begin{aligned} H_2 &= \tfrac{1}{20}\left(2 - 3\sqrt{6}\right) I_1 + \tfrac{1}{10}\left(3\sqrt{6} - 2\right) I_2 - \tfrac{3}{20}\left(3\sqrt{6} - 2\right) I_3, \\ \mathcal{H}_3 &= I_1\sqrt{I_2}\cos\phi_1, \\ \mathcal{H}_4 &= \sqrt{I_1 I_3} I_2 \cos\phi_2 + 2I_1^2 - 5I_1I_2 - I_1I_3 + 3I_2I_3, \end{aligned}$$

with $\phi_1 = 2\theta_1 + \theta_2$, $\phi_2 = \theta_1 + 2\theta_2 + \theta_3$. The resonance vectors are $k_1 = (2, 1, 0)$, $k_2 = (1, 2, 1)$, $F_1 = I_1 - 2I_2 + 3I_3$ is a formal first integral and the set S is given by $\{(2I_2 - 3I_3, I_2, I_3) \mid 0 \leq I_3 \leq 2I_2/3\}$ with $\dim S = 2$. Taking a point in the interior of S, say I^* with $I_2^* \neq 0$ and $I_2^* \neq 3I_3^*/2$, we build the function $\mathcal{H}_3(I^*, \phi_1) = (2I_2^* - 3I_3^*)\sqrt{I_2^*}\cos\phi_1$ that has a simple zero at $\phi_1^* = \pi/2$. Then part (B) of Theorem 2.1 applies and we cannot deduce stability of the origin of $\mathbb{R}^6$. In fact it is likely that the origin is unstable for the Hamiltonian system associated to (2.22), but currently none of the known theorems on instability apply. Even when stability does not hold, Theorem 2.6 applies on the action given as the first integral F_1 and the exponential time estimate is true for it.

Now we present an example of a Hamiltonian system with three degrees of freedom that has multiple resonance of order 4 and that is given by the function

$$H = H_2 + \mathcal{H}_4 + \cdots, \tag{2.23}$$

where

$$\begin{aligned} H_2 &= \tfrac{1}{20}\left(2 - 3\sqrt{6}\right) I_1 + \tfrac{3}{20}\left(3\sqrt{6} - 2\right) I_2 + \tfrac{7}{20}\left(3\sqrt{6} - 2\right) I_3, \\ \mathcal{H}_4 &= I_1^2 + I_2^2 + I_3^2 + I_1 I_2 + I_1 I_3 + I_2 I_3 + I_1^{3/2}\sqrt{I_2}\cos\phi_1 + \sqrt{I_1 I_3} I_2 \cos\phi_2, \end{aligned}$$

and $\phi_1 = 3\theta_1 + \theta_2$, $\phi_2 = \theta_1 - 2\theta_2 + \theta_3$. The resonance vectors are $k_1 = (3, 1, 0)$, $k_2 = (1, -2, 1)$. The corresponding formal first integral reads as $F_1 = -I_1 + 3I_2 + 7I_3$ whereas the set S is given by $\{(3I_1, I_1 - 7I_3, 3I_3) \mid I_1 \geq 0, I_3 \geq 0\}$, so $\dim S = 2$. Taking $I \in S$ we get

$$\mathcal{H}_4(I, \phi_1, \phi_2) = 13I_1^2 + 37I_3^2 - 23I_1I_3 + 3\sqrt{3}I_1^{3/2}\sqrt{I_1 - 7I_3}\cos\phi_1 + 3\sqrt{I_1I_3}(I_1 - 7I_3)\cos\phi_2,$$

and assuming that $I_3 \in [0, I_1/7]$, $I_1 > 0$, we know that $13I_1^2 + 37I_3^2 - 23I_1I_3 > 0$ and moreover

$$13I_1^2 + 37I_3^2 - 23I_1I_3 > |3\sqrt{3}I_1^{3/2}\sqrt{I_1 - 7I_3}| + |3\sqrt{I_1I_3}(I_1 - 7I_3)|,$$

concluding that $\mathcal{H}_4(I, \phi_1, \phi_2)$ is positive for $I \in S \setminus \{0\}$ and any $\phi_1, \phi_2 \in \mathbb{T}$. Thus, the origin of $\mathbb{R}^6$ is Lie stable for the Hamiltonian system associated to (2.23). Since $d = 1$ no Diophantine condition is needed and we can apply the bounds obtained in Theorem 2.2.

For the last example we consider the Hamiltonian which is in normal form at least including terms of order 6 given by

$$H = H_2 + \mathcal{H}_4 + \mathcal{H}_6 + \cdots \tag{2.24}$$

with

$$\begin{aligned} H_2 &= 2\sqrt{2}I_1 - 2I_2 + 4I_3 - 3\sqrt{2}I_4 + 4I_5, \\ \mathcal{H}_4 &= 3I_1^2 + 4I_5^2, \\ \mathcal{H}_6 &= 2I_3^3 + I_4^3 + 5I_5(I_3 + I_5)^2 - 2I_2^2 I_5 \sin(4\theta_2 + 2\theta_5). \end{aligned}$$

Analysing H_2 it is straightforward to deduce that M_ω is spanned by three vectors, specifically $k_1 = (0,2,0,0,1)$, $k_2 = (0,0,1,0,-1)$ and $k_3 = (3,0,0,2,0)$. From the nullspace of k_1, k_2, k_3 we build the two formal first integrals, namely $F_1 = -I_2+2I_3+2I_5$ and $F_2 = -2I_1+3I_4$. Next the set S is obtained from F_k, yielding the three-dimensional subspace of $\mathbb{R}^5$ given by $\{(3I_4, 4(I_3+I_5), 2I_3, 2I_4, 2I_5) \mid I_3, I_4, I_5 \geq 0\}$.

Considering $\mathcal{H}^4 = H_2 + \mathcal{H}_4$ we realise that it can become zero for $I \in S \setminus \{0\}$, in particular for $I^* = (0, 4I_3^*, 2I_3^*, 0, 0)$ with $I_3^* > 0$, and besides $\mathcal{H}_4 \geq 0$. Thus we need to take into account the next non-null term, that is, $\mathcal{H}^6 = H_2 + \mathcal{H}_4 + \mathcal{H}_6$. When $I \in S$ we arrive at

$$\mathcal{H}^6(I, \phi_1) = \tfrac{27}{4} I_4^2 + 4I_5^2 + 2I_3^3 + I_4^3 + I_5(I_3 + I_5)^2 \left(5 - 8\sin(2\phi_1)\right),$$

where $\phi_1 = 2\theta_2 + \theta_5$. To check that $\mathcal{H}^6$ does not change sign when I is in $S \setminus \{0\}$ it is enough to check that $\mathcal{H}_6$ does not change sign with $I_4 = I_5 = 0$. We get $\mathcal{H}_6 = 2I_3^3 > 0$ for $I = (0, 4I_3, 2I_3, 0, 0)$ with $I_3 > 0$, which is a vector in $S \setminus \{0\}$. Then $\mathcal{H}^6 \geq 0$ and $\mathcal{H}^6 = 0$ if and only if $I_3 = I_4 = I_5 = 0$. As a consequence the origin of $\mathbb{R}^{10}$ is Lie stable for the Hamiltonian system associated to H in (2.24). From the identity $H_2 = 2F_1 - \sqrt{2}F_2$ the frequency vector $(\sigma_1, \sigma_2) = (2, -\sqrt{2})$ is a Diophantine vector and Theorem 2.2 applies with $j = 6$. We emphasize that this example is not in contradiction with part (B) of Theorem 2.1 because $\mathcal{H}^4$ vanishes only on the boundary of S.

Chapter 3

Instability of equilibrium solutions of Hamiltonian systems with n degrees of freedom under the existence of resonance and an invariant ray

In this chapter we study the instability of equilibrium solutions of Hamiltonian systems with n-degrees of freedom under the existence of a single resonance, we give the details of the proof of Theorem for model and complete Hamiltonian system and we apply our main result to the case of a single resonance of order three and four. Moreover, we study the instability of equilibrium solutions of Hamiltonian systems with n-degrees of freedom under the existence of a multiple resonance of order odd. In particular for the case of resonance without interaction it is shown that the necessary conditions for instability have important simplifications and we apply our main result to examples of systems with three, four and six degrees of freedom.

Consider one autonomous Hamiltonian system with n degrees of freedom

$$\dot{\mathbf{q}} = \frac{\partial H}{\partial \mathbf{p}}, \quad \dot{\mathbf{p}} = -\frac{\partial H}{\partial \mathbf{q}}, \tag{3.1}$$

such that the origin of the phase space is an equilibrium solution. $H = H(\mathbf{q}, \mathbf{p})$ is a real analytic function of $(\mathbf{q}, \mathbf{p}) = (q_1, \ldots, q_n, p_1 \ldots, p_n)$ in a neighbourhood of the equilibrium point. It is assumed that the Taylor series of H in a neighbourhood of the origin is

$$H = H_2 + H_3 + \cdots + H_j + \cdots, \tag{3.2}$$

where H_j represents an homogeneous polynomial of degree j in $(\mathbf{q}, \mathbf{p})$, that is,

$$H_j = \sum_{|\mathbf{m}|+|\mathbf{l}|=j} h_{\mathbf{ml}}\, \mathbf{q}^{\mathbf{m}} \mathbf{p}^{\mathbf{l}}, \tag{3.3}$$

with $\mathbf{m} = (m_1, \ldots, m_n) \in \mathbb{Z}^n$, $\mathbf{l} = (l_1, \ldots, l_n) \in \mathbb{Z}^n$, $|\mathbf{m}| = |m_1| + \cdots + |m_n|$, $|\mathbf{l}| = |l_1| + \cdots + |l_n|$, $h_{\mathbf{ml}} = h_{m_1 \cdots m_n l_1 \cdots l_n}$, $\mathbf{q}^{\mathbf{m}} = q_1^{m_1} \cdots q_n^{m_n}$ and $\mathbf{p}^{\mathbf{l}} = p_1^{l_1} \cdots p_n^{l_n}$. We assume that the quadratic part is

$$H_2 = \frac{\omega_1}{2}(q_1^2 + p_1^2) + \cdots + \frac{\omega_n}{2}(q_n^2 + p_n^2), \tag{3.4}$$

where $\pm\omega_1 i, \ldots, \pm\omega_n i$ are the non null eigenvalues of the linearized system, and it is not sign-definite (so the quadratic part of H does not decide the nonlinear stability of the point $(0, 0)$).

3.1 Single resonance

In order to enunciate and to prove our main results, we assume that the Hamiltonian system associated to (3.2) is stable in the linear approximation and possesses a single vector $\mathbf{k}$ of resonance (i.e., $\mathbf{k}{\cdot}\omega = 0$) of order $s = |\mathbf{k}| > 2$, without resonance of lower order, but it may exist resonance of upper order. It is known from Theorem 3.1-(c) in [26], that a necessary condition in order to have instability of the origin under the assumption of type of resonance is that $k_1, k_2, \ldots, k_n \geq 0$. Thus, here we assume that $\mathbf{k} = (k_1, \ldots, k_u, k_{u+1}, \ldots, k_n)$ where $k_i > 0$ if $i = 1, \ldots, u$ and $k_j = 0$ if $j = u + 1, \ldots, n$.

Next, we normalize the Hamiltonian function H given in (3.2) in Lie normal form up to order s. Thus, after introducing the action-angle coordinates by $(q, p) \to (I, \varphi)$, $I = (I_1, \ldots, I_n)$, $\varphi = (\varphi_1, \ldots, \varphi_n)$ where

$$q_j = \sqrt{2I_j}\cos\varphi_j, \quad p_j = \sqrt{2I_j}\sin\varphi_j,\ j = 1, \ldots, n \tag{3.5}$$

the normalized Hamiltonian function H (or simply the complete Hamiltonian function) up to order s (i.e., the Lie process of normalization is finite and it is performing until order s) can be written in action-angle variables as

$$H = H(\mathbf{I}, \varphi) = H_2(\mathbf{I}) + \cdots + H_{2l}(\mathbf{I}) + H_s(\mathbf{I}, \phi) + \mathcal{R},$$

where $2l$ is less than the natural number s, $\phi = \mathbf{k} \cdot \varphi = k_1\varphi_1 + \cdots + k_n\varphi_n$ and $\mathcal{R} = \mathcal{R}(\mathbf{I}, \varphi) = O(I^{(s+1)/2})$. Note that the expression $H_s(\mathbf{I}, \phi)$ contains resonant terms of order s and it is characterized by

$$H_s(\mathbf{I}, \phi) = H_s^0(\mathbf{I}) + H_s^1(\mathbf{I})\cos\phi, \tag{3.6}$$

where $H_s^0(\mathbf{I}) = \sum_{\mu_1+\cdots+\mu_n=s/2} a_{\mu_1\ldots\mu_n} \prod_{i=1}^{n} I_i^{\mu_i}$ ($\mu_i \in \mathbb{Z}_+ \cup \{0\}$) when s is even; $H_s^0(\mathbf{I}) = 0$ when s is odd and $H_s^1(\mathbf{I}) = 2A\prod_{i=1}^{u} I_i^{k_i/2}$ (similar analysis has been performed in [27] and [28]).

Next, we will denote by

$$H^s = H^s(\mathbf{I},\phi) = H_2(\mathbf{I}) + H_4(\mathbf{I}) + \cdots + H_{2l}(\mathbf{I}) + H_s(\mathbf{I},\phi), \tag{3.7}$$

the model or the truncated Hamiltonian function, where H_s is defined in (3.6).

For the model Hamiltonian system (3.7) we have the following result.

Theorem 3.1. *Under the conditions $k_1, k_2, \ldots, k_n \geq 0$, $H_{2j}(k) = 0$, for all $j = 2, \ldots, l$ and*

$$|A| > \frac{\left| \sum_{\mu_1+\cdots+\mu_n=s/2} a_{\mu_1\ldots\mu_n} \prod_{j=1}^{u} k_j^{\mu_j} \right|}{2\prod_{i=1}^{u} k_i^{k_i/2}}, \tag{3.8}$$

the model Hamiltonian system associated to (3.7) *possesses an invariant ray type solution. In particular, the origin associated to the model Hamiltonian system (3.7) is unstable in the Liapunov sense.*

The proof of this theorem can be found in Section 3.1.1. For the complete Hamiltonian system our main result is the following.

Theorem 3.2. *The equilibrium solution* $(0,0)$ *of the complete Hamiltonian system associated to*

$$H^s = H^s(\mathbf{I},\phi) = H_2(\mathbf{I}) + H_s(\mathbf{I},\phi) + \mathcal{R}(\mathbf{I},\varphi), \tag{3.9}$$

(i.e., for any type of perturbation function $\mathcal{R}(\mathbf{I},\varphi) = O(I^{(s+1)/2})$) is unstable in the Liapunov sense, whenever $k_1, k_2, \ldots, k_n \geq 0$, $H_{2j}(I) \equiv 0$, for all $j = 2, \ldots, l$ and inequality (3.8) holds.

The proof of this theorem can be found in Section 3.1.2. As we will see during the proof when the order of the resonance is $s = 4$, we can weaken the condition $H_4(I) \equiv 0$, in fact it is enough to assume that $H_4(k) = 0$. But for $s > 4$ we need the assumptions $H_{2j}(I) \equiv 0$, for all $j = 2, \ldots, l$.

In [26] the case of a single resonance was considered. However, looking at the arguments of the proof, we observe that the arguments used during the proof need the requirement of existence of a single resonance up to order $2s-2$, that is, up to the order or resonance s. In fact, this supposition is very important in order to control the upper terms in the use of the classic Chetaev Theorem (i.e., the existence of a differentiable function V which is negative with negative derivative on the region $V<0$). Note that in our Main Theorem 3.2 we require the existence of a single resonance up to order s, i.e., it is permitted to have any other resonance of upper order. The arguments in our proof use essentially and strongly the existence of the invariant ray solution of the Hamiltonian system model, and the introduction of convenient coordinates called $(R,\phi,\theta_2,\ldots,\theta_n)$ which permit us to control the point $\mathbf{I}=(0,\ldots,0)$ by a unique variable called $R=0$. Also, we obtain an uncoupled angular system (with $R=0$) such that the invariant ray solution in the angular coordinates $(\phi,\theta_2,\ldots,\theta_n)$ is an equilibrium which is asymptotically stable. Taking the Liapunov function given by the previous fact, we are able to construct a convenient generalized cone K such that we can apply Chetaev's Theorem 1.32 enunciated in the Appendix.

3.1.1 Proof of Theorem 3.1

The Hamiltonian system model associated to (3.7) (that is, the differential equations derived from (3.10)) assumes the form

$$\begin{aligned}
\dot{I}_j &= -2Ak_j\prod_{i=1}^{u} I_i^{k_i/2}\sin\phi,\ \text{for } j=1,\ldots,u,\\
\dot{I}_j &= 0,\ \text{for } j=u+1,\ldots,n,\\
\dot{\phi} &= -\sum_{j=1}^{n}k_j\left(\frac{\partial H_4(I)}{\partial I_j}+\cdots\frac{\partial H_{2l}(I)}{\partial I_j}\right)-A\prod_{i=1}^{u}I_i^{k_i/2}\sum_{j=1}^{n}\frac{k_j^2}{I_j}\cos\phi-\\
&\sum_{\mu_1+\cdots+\mu_n=s/2}a_{\mu_1\ldots\mu_n}\prod_{j=1}^{n}I_j^{\mu_j}\sum_{j=1}^{n}\frac{\mu_jk_j}{I_j}.
\end{aligned}\tag{3.10}$$

Note that the functions $F_j=k_jI_1-k_1I_j$, with $j=2,\ldots,n$ are first integrals of the Hamiltonian model defined by the function in (3.7). In fact, derivating F_j through the solutions of the Hamiltonian system model (3.7), we get

$$\begin{aligned}
\dot{F}_j &= k_j\dot{I}_1-k_1\dot{I}_j=k_j\frac{\partial H^s}{\partial\varphi_1}-k_1\frac{\partial H^s}{\partial\varphi_j}\\
&= -Ak_jk_1\prod_{j=1}^{u}I_j^{k_j/2}\sin\phi+Ak_1k_j\prod_{j=1}^{u}I_j^{k_j/2}\sin\phi\\
&= 0.
\end{aligned}$$

We propose an unbounded and increasing invariant ray type solution for the Hamiltonian system (3.10) on the invariant surface $F_j = 0$ with $j = 2, \ldots, n$. Let

$$F_j(t) = \frac{k_j}{k_1}\sigma(t),\ j = 2, \ldots n, \quad \phi = \phi_*, \tag{3.11}$$

where $\sigma(t) = I_1 > 0$, for all t and ϕ_* is a constant, which is chosen such that it is an invariant ray solution. In fact, replacing these expressions in the system (3.10) and as by hypothesis $H_{2j}(k) = 0$ for $j = 1, \ldots, l$, we arrive to

$$\begin{aligned}
\dot{r}_j &= -2A\ k_j\ k_1^{-s/2}\prod_{i=1}^{u} k_i^{k_i/2}\sigma^{s/2}\sin\phi_*, \\
\dot{\phi} &= -s\left(\frac{\sigma}{k_1}\right)^{s/2-1}\left(\frac{1}{2}\sum_{\mu_1+\cdots+\mu_n=s/2} a_{\mu_1\ldots\mu_n}\prod_{j=1}^{u}k_j^{\mu_j} + A\prod_{i=1}^{u}k_i^{k_i/2}\cos\phi_*\right).
\end{aligned}$$

As ϕ_* is constant, then $\dot{\phi} = 0$, which is equivalent to have

$$\cos\phi_* = -\frac{\displaystyle\sum_{\mu_1+\cdots+\mu_n=s/2} a_{\mu_1\ldots\mu_n}\prod_{j=1}^{u}k_j^{\mu_j}}{\displaystyle 2A\prod_{i=1}^{u}k_i^{k_i/2}}.$$

Note that ϕ_* is well defined, because by hypothesis (3.8) $|\cos\phi_*| < 1$ and $A \neq 0$ in any case (i.e., s odd or even), and we call the attention that for the case s odd, we have $\phi_* = \pm\frac{\pi}{2}$. Moreover, as $I_1(t) = \sigma(t)$, then $\dot{I}_1(t) = \dot{\sigma}(t)$, that is,

$$\dot{\sigma}(t) = -2Ak_1^{-s/2+1}\prod_{i=1}^{u}k_i^{k_i/2}\sigma(t)^{s/2}\sin\phi_*, \tag{3.12}$$

we must choose ϕ_* such that $A\sin\phi_* < 0$, i.e., $\sin\phi_* > 0$ for $A < 0$ and $\phi_* + \pi$ such that $\sin\phi_* < 0$ for $A > 0$. Also, it is verified that $\dot{I}_j$ is coherent because $\dot{I}_j = \frac{k_j}{k_1}\dot{\sigma}(t)$, so it is verified that $\dot{\sigma} > 0$. Note that the general solution of (3.12) is

$$\sigma(t) = \frac{1}{\displaystyle -c + 2Atk_1^{-s/2+1}\prod_{i=1}^{u}k_i^{k_i/2}\sin\phi_*}.$$

Therefore, there exists an invariant ray type solution for the Hamiltonian system model (3.10), in particular, the origin of this system is unstable in the Liapunov sense. ∎

3.1.2 Proof of Theorem 3.2

We are going to prove the instability in the Liapunov sense of the origin of the complete Hamiltonian system, that is, when we introduce the perturbed terms in the Hamiltonian function model (3.7).

First we introduce the convenient change of variables (which is not symplectic) R, θ_j, ϕ and the new time τ, in a neighborhood of the invariant ray solution of the system (3.10), such that now the origin corresponds to $R = 0$. These new coordinates are given by

$$\begin{aligned} I_1 &= R, \\ I_j &= \tfrac{k_j}{k_1} R(1+\theta_j), \quad \text{for } j = 2, \dots, u, \\ I_j &= I_1 \theta_j, \quad \text{for } j = u+1, \dots, n, \\ d\tau &= R^{s/2-1} dt. \end{aligned} \tag{3.13}$$

As $H_{2j}(I) \equiv 0$ for $j = 2, \dots, l$, and after some manipulations we arrive that the complete Hamiltonian system associated to (3.9) in these new variables assumes the form

$$\begin{aligned} \tfrac{dR}{d\tau} &= R f_0(\phi, \theta) + \mathcal{O}(R^{3/2}), \\ \tfrac{d\phi}{d\tau} &= f_1(\phi, \theta) + \mathcal{O}(R^{1/2}), \\ \tfrac{d\theta_j}{d\tau} &= f_j(\phi, \theta) + \mathcal{O}(R^{1/2}), \quad j = 2, \dots, n, \end{aligned} \tag{3.14}$$

where

$$\begin{aligned} f_0(\phi,\theta) &= -2Ah_0(\theta)\sin\phi, \\ f_1(\phi,\theta) &= -Ah_1(\theta)\cos\phi - h_2(\theta), \\ f_j(\phi,\theta) &= 2Ah_0(\theta)\theta_j\sin\phi, \\ h_0(\theta) &= k_1^{(s-k_1)/2+1}\prod_{i=2}^{u}[k_i(1+\theta_i)]^{k_i/2}, \\ h_1(\theta) &= k_1^{(s-k_1)/2+1}\prod_{i=2}^{u}[k_i(1+\theta_i)]^{k_i/2}\left(k_1 + \sum_{i=2}^{u}\frac{k_i}{1+\theta_i}\right), \\ h_2(\theta) &= \sum_{\mu_1+\cdots+\mu_n=s/2} a_{\mu_1\dots\mu_n} k_1^{\alpha}\prod_{i=2}^{u}[k_i(1+\theta_i)]^{\mu_i}\prod_{i=u+1}^{n}\theta_i^{\mu_i}\left(\mu_1 + \sum_{j=2}^{u}\frac{\mu_j}{1+\theta_j}\right), \end{aligned}$$

with $\alpha = 1 - \sum_{j=2}^{u}\mu_j$. Note that $h_0(\theta) > 0,\ h_1(\theta) > 0$. From now on, we will call as by angular system, the ordinary differential equations associated to $\frac{d\phi}{d\tau}$ and $\frac{d\theta_j}{d\tau}$ with $j = 2, \ldots, n$ in the system (3.14), making $R = 0$. This system is simply

$$\begin{aligned} \tfrac{d\phi}{d\tau} &= -Ah_1(\theta)\cos\phi - h_2(\theta), \\ \tfrac{d\theta_j}{d\tau} &= 2Ah_0(\theta)\theta_j \sin\phi, \quad j = 2, \ldots, n. \end{aligned} \tag{3.15}$$

We call the attention that for resonance of order 4, we can only assume that $H_4(k) = 0$. But, when the order of resonance is upper than 4, we need the hypothesis $H_{2j}(I) \equiv 0$ because the angular system in (3.14) cannot be decoupled of the variable R.

Next, we point out that the invariant ray solution of the system (3.10) now corresponds to an equilibrium point of the angular system (3.15). In fact, the invariant ray solution is of the form

$$I_j(t) = \frac{k_j}{k_1}\sigma(t), \quad \phi_* = \arccos\left(-\frac{\displaystyle\sum_{\mu_1+\cdots+\mu_n=s/2} a_{\mu_1\ldots\mu_n}\prod_{j=1}^{u} k_j^{\mu_j}}{\displaystyle 2A\prod_{i=1}^{u} k_i^{k_i/2}}\right) \tag{3.16}$$

with $j = 2, \ldots, u$ and $A\sin\phi_* < 0$. Moreover, in the new coordinates (3.13) we must have

$$I_j = \frac{k_j}{k_1}R(1+\theta_j),\ j = 2, \ldots, u,\ I_j = R\theta_j,\ j = u+1, \ldots, n,\ R(t) = \sigma(t),$$

and so $\theta_j = 0$ for $j = 2, \ldots, n$. On the other hand, the point

$$(\phi_*, \theta_2^*, \theta_3^*, \theta_4^*, \ldots, \theta_n^*) = \left(\arccos\left(-\frac{\displaystyle\sum_{\mu_1+\cdots+\mu_n=s/2} a_{\mu_1\ldots\mu_n}\prod_{j=1}^{u} k_j^{\mu_j}}{\displaystyle 2A\prod_{i=1}^{u} k_i^{k_i/2}}\right), 0, 0, 0, \ldots, 0\right),$$

is an equilibrium solution of the angular system (3.15). Because $h_2(0) = 0$. Now, we are going to study the type of stability of this equilibrium point associated to the angular system. Linearising the angular system (3.15), we obtain the matrix

$$M = \begin{pmatrix} \frac{\partial f_1}{\partial \phi} & \frac{\partial f_1}{\partial \theta_2} & \cdots & \frac{\partial f_1}{\partial \theta_n} \\ \frac{\partial f_2}{\partial \phi} & \frac{\partial f_2}{\partial \theta_2} & \cdots & \frac{\partial f_2}{\partial \theta_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial \phi} & \frac{\partial f_n}{\partial \theta_2} & \cdots & \frac{\partial f_n}{\partial \theta_n} \end{pmatrix},$$

where $f_1 = \dfrac{d\phi}{d\tau}$, $f_j = \dfrac{d\theta_j}{d\tau}$, for $j = 2, \ldots, n$. Evaluating M at the equilibrium point $(\phi_*, 0, 0, 0, \ldots, 0)$, we have

$$M|_{(\phi_*,0,\ldots,0)} = \begin{pmatrix} Ah_1(0)\sin\phi_* & \frac{\partial f_1}{\partial\theta_2}(\phi_*,0,\ldots,0) & \ldots & \frac{\partial f_1}{\partial\theta_n}(\phi_*,0,\ldots,0) \\ 0 & 2Ah_0(0)\sin\phi_*, & \ldots & \frac{\partial f_2}{\partial\theta_n}(\phi_*,0,\ldots,0) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 2Ah_0(0)\sin\phi_*. \end{pmatrix}.$$

It is verified easily that the eigenvalues of the above matrix are

$$\begin{aligned} \lambda_1 &= Ah_1(0)\sin\phi_* = sAk_1^{(s-1)/2+1}\prod_{i=2}^{u} k_i^{k_i/2}\sin\phi_*, \\ \lambda_2 &= 2Ah_0(0)\sin\phi_* = 2Ak_1^{(s-1)/2+1}\prod_{i=2}^{u} k_i^{k_i/2}\sin\phi_*, \end{aligned}$$

with λ_2 of multiplicity $n-1$. As by hypothesis $A\sin\phi_* < 0$, we have that all the eigenvalues are negative. Therefore, the equilibrium point $\phi = \phi_*$, $\theta_j = 0$, $j = 2, \ldots, n$ (and $R = 0$) of the angular system (3.15) is asymptotically stable. By virtue of Theorem 1.34 in Appendix, in a small neighborhood of the equilibrium point $(\phi_*, 0, \ldots, 0)$ there exists a Liapunov function, called $l = l(\phi, \theta_2, \ldots, \theta_n)$ which is positive definite with negative definite derivative along the solutions of the angular system (3.15).

Now, we pass to analyze the instability of the origin associated to the complete system (3.14) using the information obtained for the angular system (3.15).

We consider that on the surface $l = l(\phi, \theta_2, \ldots, \theta_n) = c$, there exist convenient constants $c > 0$ sufficiently small and $\gamma_1 > 0$ (to be chosen conveniently later) satisfying

$$\frac{dl}{d\tau}|_{(3.15)} \leq -\gamma_1 < 0, \tag{3.17}$$

where $\frac{dl}{d\tau}|_{(3.15)}$ means the derivative of the function l with respect to τ through the solutions of the vector field (3.15). It is assumed that c is sufficiently small such that

$$Ah_0(\theta)\sin\phi \leq -\gamma_0 < 0, \tag{3.18}$$

in the region $l = l(\phi, \theta_2, \ldots, \theta_n) \leq c$, for some convenient constant $\gamma_0 > 0$.

Next, we define the generalized "cone" K on the phase space, by

$$K = \{(R, \phi, \theta) : \; l(\phi, \theta) \leq c, \quad R \leq R_0\},$$

for R_0 convenient and sufficiently small (to be chosen conveniently later). See Figure 3.1 for a representation of the set K. Note that the invariant ray solution starts on this set.

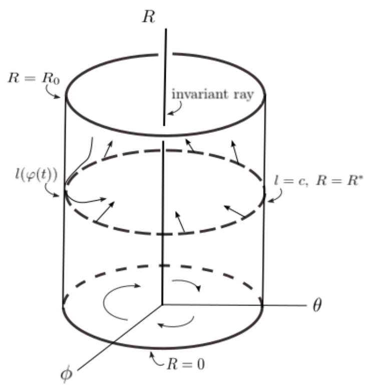


Figure 3.1: Representation of the cone $K = \{(R, \phi, \theta) :\ l(\phi, \theta) \leq c,\ R \leq R_0\}$.

In order to study the instability of the origin of the system (3.14) we are going to apply Chetaev's Theorem 1.32 from the Appendix. We take as Chetaev function $V(R, \phi, \theta) = R$, our aim is to prove that the derivative of V through the solutions of the system (3.14) is positive definite in the interior of the cone K; and that the flow of the system (3.14) in the interior of the cone K is invariant, i.e., the flow of the system (3.14) cannot go through the lateral surfaces of the cone, and the set $R = 0$ is invariant for the system (3.14).

Let $\varphi(t) = (R(t), \phi(t), \theta_2(t), \ldots, \theta_n(t))$ be a solution of the system (3.14) with initial condition in the interior of the set K. First, we observe that by continuity of the flow with respect to initial conditions and small perturbation vector field (3.14) and the vector field (3.15), it follows by (3.17) that for R sufficiently small

$$\begin{aligned}
\frac{dl}{d\tau}|_{(3.14)} &= \frac{dl}{d\tau}|_{(3.15)} + \mathcal{O}(R^{1/2}) \\
&\leq -\gamma_1 + \mathcal{O}(R^{1/2}) \\
&= -\frac{1}{2}\gamma_1 + \left(-\frac{1}{2}\gamma_1 + \mathcal{O}(R^{1/2})\right) \\
&\leq -\frac{1}{2}\gamma_1 < 0,
\end{aligned}$$

whether we take $R \leq R_1$ such that $-\frac{1}{2}\gamma_1 + \mathcal{O}(R^{1/2}) < 0$. On the other hand, by the first equation in (3.15) it is clear that the set $R = 0$ is invariant by the flow associated to (3.14), in particular, the inferior surface of the cone $l \leq c$ and $R = 0$ is invariant by the flow of the system (3.15). Thus, we have proved that if a solution of (3.14) starting in the cone K touches the lateral surface of the cone, then this cannot leave the cone. Since, $V = R > 0$, we need to compute $\frac{dV}{d\tau}$ through the solutions of (3.14) which remain in the interior of K for all time. From (3.18), we have

$$\begin{aligned}
\frac{dV}{d\tau} &= -2ARh_0(\theta)\sin\phi + \mathcal{O}(R^{3/2}) \\
&\geq \gamma_0 R^2 + \mathcal{O}(R^{3/2}) \\
&= \frac{1}{2}\gamma_0 R + R\left(\frac{1}{2}\gamma_0 + \mathcal{O}(R^{1/2})\right) \\
&\geq \frac{1}{2}\gamma_0 R > 0,
\end{aligned}$$

choosing $R \leq R_2$ such that $\frac{1}{2}\gamma_0 + \mathcal{O}(R^{1/2}) > 0$. Therefore, taking $R_0 = \min\{R_1, R_2\}$ we are in condition to apply Chetaev's Theorem 1.32 to the equilibrium point $(0,0)$ of the system (3.14) which guarantees the instability in the Liapunov sense. ∎

3.1.3 Remarks on particular cases

In the case where the system (3.1) possesses a single resonance up to order 3, the vector of resonance (reordering if it is necessary) has the following two possibilities: $k = (2, 1, 0, \ldots, 0)$ and $k = (1, 1, 1, 0, \ldots, 0)$. Here the normalized (Lie normal form) of the truncated Hamiltonian (or simply, Hamiltonian model) up to order three (in cartesian coordinates) in action-angles variables assumes the form

$$H^3 = \sum_{i=1}^{n} \omega_i I_i + 2A \prod_{i=1}^{u} I_i^{k_i/2} \cos\phi, \tag{3.19}$$

where $\phi = k \cdot \varphi$. Therefore, the condition of existence of an invariant ray is simply $A \neq 0$.

For the case where the system (3.1) possesses a single resonance up to order 4, the vector of resonance (reordering if it is necessary) has the following three possibilities: $k = (3, 1, 0, \ldots, 0)$, $k = (2, 1, 1, 0, \ldots, 0)$ and $k = (1, 1, 1, 1, 0, \ldots, 0)$. Thus the normalized (Lie normal form) of the truncated Hamiltonian (or simply, Hamiltonian model) up to order four (in cartesian coordinates) in action-angles variables assumes the form

$$H^4 \;=\; \sum_{i=1}^{n}\omega_i I_i + \sum_{1\le i\le j\le n} a_{ij} I_i I_j + 2A\prod_{i=1}^{u} I_i^{k_i/2}\cos\phi, \tag{3.20}$$

where $\phi = k\cdot\varphi$. Therefore, the condition of existence of an invariant ray is equivalent to

$$\left|\sum_{1\le i\le j\le u} a_{ij}k_i k_j\right| < 2|A|\prod_{i=1}^{u} k_i^{k_i/2}.$$

3.2 Multiple resonance

In order to enunciate and to prove our main results, we assume that the frequency vector $\omega = (\omega_1,\ldots,\omega_n)$ admits multiple resonances of the same order and this is odd. The vectors of resonance are denoted by $k_1,\ldots,k_\mu$ which are linearly independent over $\mathbb{Q}$, thus

$$\begin{aligned} M_\omega &= \{k_j = (k_{j1},k_{j2},\ldots,k_{jn})\in\mathbb{Z}^n;\; k_j\cdot\omega = k_{j1}\omega_1+\ldots+k_{jn}\omega_n = 0\}\\ &= k_1\mathbb{Z}+\ldots+k_\mu\mathbb{Z}, \end{aligned} \tag{3.21}$$

such that, the vectors of resonance have the same odd order, that is, $|k_1| = |k_2| = \ldots = |k_\mu| = s = 2m+1$ and denote by q the maximum number of non-zero components among the μ vectors of resonance.

To investigate the behavior of system (3.1) near the equilibrium solution $(0,0)$, it is convenient in our approach to put the Hamiltonian function H given in (3.2) in its Lie normal form up to order s. Introducing the action-angle coordinates as in (3.5) and we have that the Lie normalization of Hamiltonian function H in (3.2) can be written in action-angle variables as

$$H = H(I,\varphi) = H^s(I,\varphi) + \tilde{\mathcal{H}}(I,\varphi), \tag{3.22}$$

where

$$H^s = H^s(I,\varphi) = H_2(I) + H_4(I) + \ldots + H_{s-1}(I) + H_s(I,\varphi), \tag{3.23}$$

where

$$\begin{aligned} H_2 &= \sum_{j=1}^{n} \omega_j I_j, \\ H_l(I) &= \sum_{m_1+\cdots+m_n=l/2} a_{m_1\ldots m_n} \prod_{i=1}^{n} I_i^{m_i},\ l=4,\ldots,s-1, \\ H_s(I,\varphi) &= \sum_{\nu=1}^{\mu} H_s^{\nu}(I,k_\nu\cdot\varphi), \\ H_s^{\nu}(I,k_\nu\cdot\varphi) &= 2A_\nu \prod_{l=1}^{q} I_l^{|k_{\nu l}|/2}\cos(k_\nu\cdot\varphi),\ for\ \nu=1,\ldots,\mu, \end{aligned} \tag{3.24}$$

and $\tilde{\mathcal{H}}(I,\varphi)$ represents higher order terms in $I^{(s+1)/2}$ not necessarily normalized. This characterization can be found in [28]. In our analysis we assume that $H_{2j}(I)=0$, for all $j=2,\ldots,(s-1)/2$, because if one the $H_{2j}\neq 0$ for some $j=2,\ldots,(s-1)/2$ then according [28] the null solution of (3.22) is Lie-stable and formally stable. Next, we denote by

$$H^s = H^s(\mathbf{I},\varphi) = H_2(\mathbf{I}) + H_s(\mathbf{I},\varphi), \tag{3.25}$$

the model or the truncated Hamiltonian function, where H_s is defined in (3.24).

Initially, for the model Hamiltonian system (3.25) we have the following result.

Theorem 3.3. *Under the above assumptions and assuming further that the system*

$$c_j = \sum_{\eta=\mu_0+1}^{\mu} |A_\eta| k_{\eta j} \prod_{l=1}^{q} c_j^{|k_{\eta l}|/2} - \sum_{\zeta=1}^{\mu_0} |A_\zeta| k_{\zeta j} \prod_{l=1}^{q} c_j^{|k_{\zeta l}|/2}, \quad j=1,\ldots,q \tag{3.26}$$

has solution $c_1,\ldots,c_q\in\mathbb{R}^+$, then the Hamiltonian system model (3.23) has an invariant ray type solution $\theta_\zeta^0 = \frac{\pi}{2}\frac{A_\zeta}{|A_\zeta|}$, for $A_\zeta\neq 0$ with $\zeta=1,\ldots,\mu_0$ (if the components of k_ζ change sign) and $\theta_\eta^0 = -\frac{\pi}{2} sign(A_\eta)$, for $A_\eta\neq 0$ with $\zeta=\mu_0+1,\ldots,\mu$ (if the components of k_η do not change sign), $0\le\mu_0\le\mu$. In particular, the origin of the Hamiltonian system model (3.23) is unstable in the Liapunov sense.

The proof of this theorem can be found in Section 3.2.1. In Corollary 3.5 are characterized the conditions of the previous result for the case of vector of resonances without interaction. Here is observed that the conditions (3.26) are simplified an enormity.

Now, we introduce the following notation

$$\begin{aligned}
B_{\beta h} &= \sum_{\nu=1}^{\mu} \frac{2R_\nu^0}{(q-\beta+1)\sqrt{q-\beta}} \left(\sum_{j=\beta+1}^{q} \frac{Q_{\nu j}^0}{c_j} - \frac{Q_{\nu\beta}^0}{c_\beta}(q-\beta) \right) \frac{1}{2\sqrt{q-h}} \times \\
&\quad \left(\sum_{l=h+1}^{q} |k_{\nu l}| - |k_{\nu h}|(q-h) \right) - 2\delta_{\beta h}, \\
B_{\beta,n+\nu} &= 0, \quad B_{n+\nu,n+i} = -R_i^0 \sum_{j=1}^{q} \frac{|k_{\nu j}| Q_{ij}^0}{c_j}, \quad B_{n+\nu,\beta} = 0, \\
R_\nu^0 &= \prod_{j=1}^{q} c_j^{|k_{\nu j}|/2}, \quad Q_{\nu j}^0 = Q_{\nu j}(\theta_\nu^0),
\end{aligned}$$

with $\beta,\ h = 1,\ldots,q$ and $i,\ \nu = 1,\ldots,\mu$. For the complete Hamiltonian system (3.22) our main result is the following.

Theorem 3.4. *Let $\gamma \in \mathbb{N} \cup \{0\}$ such that*

$$\gamma = \left[\max_{\substack{\beta \in \{1,\ldots,q-1\} \\ \nu \in \{1,\ldots,\mu\}}} \left\{ \frac{1}{2} \sum_{h=1}^{q-1} B_{\beta h} - 1, \frac{1}{2} \sum_{i=1}^{\mu} B_{n+\nu,n+i} - 1 \right\} \right],$$

here $[\,]$ means integer part. Under the conditions of Theorem 3.3 and assuming that $\det(B_{\nu\zeta} - NI) \neq 0$, $N = 1,2,\ldots,2(1+\gamma)$ $(\nu,\zeta = 1,\ldots,n+\nu; \nu,\zeta \neq q,\ldots,n)$ (I is the identity matrix), the origin of the complete Hamiltonian system (3.22) is unstable in the Liapunov sense.

The proof of this theorem can be found in Section 3.2.2. Important simplifications of the previous theorem are point out in Corollary 3.11 in the the case of vector of resonances without interactions.

For the case of multiple resonances of the same order and even, it can be seen that in the Hamiltonian H_s^ν in its Lie normal form is as follows

$$H_s^\nu = \sum_{m_1+\cdots+m_n=s/2} a_{m_1\ldots m_n} \prod_{i=1}^{n} I_i^{m_i} + 2A_\nu \prod_{l=1}^{q} I_l^{k_{\nu l}/2} \cos(k_\nu \cdot \varphi),$$

for $\nu = 1,\ldots,\mu$ and $m_i \in \mathbb{Z}_+ \cup \{0\}$. In this case the characterization of the invariant ray solution is too complicated, and for instance it is a work in progress.

In Khazin [42] the case of two resonance vectors of order three was considered. He proved the instability of the origin. The case where the resonance vectors have interaction in two frequencies was analyzed in [43] for the particular vectors of resonance $k_1 = (2,1,0)$ and $k_2 = (-1,1,1)$. Here the corresponding Hamiltonian function in its Lie normal form is

$$H = I_1 - 2I_2 + 3I_3 + 2A_1\sqrt{I_1^2 I_2}\cos(2\varphi_1+\varphi_2) + 2A_2\sqrt{I_1 I_2 I_3}\cos(-\varphi_1+\varphi_2+\varphi_3) + \dots.$$

He proved the instability of the null solution of the complete Hamiltonian system, under the existence of an invariant ray of the model Hamiltonian system, that is, when $|A| \geq 1$ with $A = A_1/|A_2|$. For the case $|A| < 1$ the proof of the instability of the origin was given in [43]. In [74] and [72] the instability of the origin under the presence of μ resonances of the same odd order for an ordinary differential system was studied, for a model and complete system, respectively. In a later work [73] Zhavnerchik treated the case of multiple resonance of odd order also for ordinary differential systems (ODE). According to our point of view the author omitted important details in the demonstration of the main result and the importance of the assumptions are not clear. In this work the author mentioned a small comment in reference to the Hamiltonian case, but no details are given of the possible simplifications of the demonstration and the technical conditions of the result in the case that the ODE has a Hamiltonian structure. In fact, taking into account the Hamiltonian structure we note that here the conditions are less restrictive than the ones in [73]. Moreover, in or approach we perform a detailed analysis of both the existence of the invariant ray, the different types of coordinates, and the proper use of Chetaev's Theorem 1.33 on a generalized cone. In [28] it was studied the instability of the null solution of the complete Hamiltonian system in the presence of μ resonances of the same order s (regardless of whether the order is even or odd), but there is no other type of resonance up to $2s$ inclusively, without interaction. Here in order to apply the classical Chetaev Theorem in a convenient connected component, it is necessary to control the perturbing terms, for which reason the Hamiltonian must be normalized up to order $2s$. In the present work we only need to normalize up to order of the resonance and the result is valid as with and without interactions between the vector of resonances.

3.2.1 Proof of Theorem 3.3

The model Hamiltonian system associated to the Hamiltonian function (3.24) is

$$\begin{aligned}
\dot{I}_j &= -2\sum_{\nu=1}^{\mu} A_\nu k_{\nu j}\prod_{l=1}^{q} I_l^{|k_{\nu l}|/2}\sin(k_\nu\cdot\varphi), \quad j=1,\dots,q,\\
\dot{I}_\alpha &= 0, \quad \alpha=q+1,\dots,n\\
\dot{\varphi}_j &= -\omega_j-\sum_{\nu=1}^{\mu}\frac{1}{I_j}A_\nu|k_{\nu j}|\prod_{l=1}^{q} I_l^{|k_{\nu l}|/2}\cos(k_\nu\cdot\varphi), \quad j=1,\dots,n.
\end{aligned} \tag{3.27}$$

Considering the coordinates associated the vector of resonances $\theta_\nu = k_\nu\cdot\varphi = \sum_{j=1}^{q} k_{\nu j}\varphi_j$, we have that

$$\dot{\theta}_\nu = -\sum_{j=1}^{q}\sum_{\nu=1}^{\mu}\frac{1}{I_j}A_\nu|k_{\nu j}|k_{\nu j}\prod_{l=1}^{q} I_l^{|k_{\nu l}|/2}\cos(\theta_\nu), \quad \nu=1,\dots,\mu, \tag{3.28}$$

and introducing the notations

$$R_\nu=\prod_{l=1}^{q} I_l^{|k_{\nu l}|/2}, \quad Q_{\nu j}(\theta_\nu)=-A_{\nu j}k_{\nu j}\sin(\theta_\nu), \quad Q'_{\nu j}(\theta_\nu)=\frac{dQ_{\nu j}}{d\theta_\nu}=-A_{\nu j}k_{\nu j}\cos(\theta_\nu),$$

the model Hamiltonian system (3.27) reads as

$$\begin{aligned}
\dot{I}_j &= 2\sum_{\nu=1}^{\mu} R_\nu Q_{\nu j}(\theta_\nu), \quad j=1,\dots,q, \quad \dot{I}_\alpha=0, \quad \alpha=q+1,\dots,n\\
\dot{\theta}_\nu &= \sum_{j=1}^{q}\sum_{\nu=1}^{\mu}|k_{\nu j}|\frac{1}{I_j}R_\nu Q'_{\nu j}(\theta_\nu), \quad \nu=1,\dots,\mu.
\end{aligned} \tag{3.29}$$

We propose an unbounded and increasing invariant type solution for the model Hamiltonian system (3.29) such that

$$\begin{aligned}
I_j &= c_j b(t), \quad c_j>0,\ j=1,\dots,q, \quad \dot{b}=2b^{s/2},\\
\theta_\nu^0 &= \begin{cases} \frac{\pi}{2}\frac{A_\nu}{|A_\nu|}, \text{ if the components of } k_\nu \text{ change sign,}\\ \qquad \text{for all } \nu=1,\dots,\mu_0\\ -\frac{\pi}{2}\frac{A_\nu}{|A_\nu|}, \text{ if the components of } k_\nu \text{ do not change sign,}\\ \qquad \text{for all } \nu=\mu_0+1,\dots,\mu. \end{cases}
\end{aligned} \tag{3.30}$$

In fact, replacing these expressions in system (3.29), we arrive at

$$\dot{I}_j = -2b(t)^{s/2}\left[\sum_{\nu=1}^{\mu_0}|A_\nu|k_{\nu j}\prod_{l=1}^{q}c_l^{|k_{\nu l}|/2} - \sum_{\nu=\mu_0+1}^{\mu}|A_\nu|k_{\nu j}\prod_{l=1}^{q}c_l^{|k_{\nu l}|/2}\right].$$

As $I_j = c_j b(t)$, then comparing with the previous equation we take

$$c_j = -\sum_{\nu=1}^{\mu_0}|A_\nu|k_{\nu j}\prod_{l=1}^{q}c_l^{|k_{\nu l}|/2} + \sum_{\nu=\mu_0+1}^{\mu}|A_\nu|k_{\nu j}\prod_{l=1}^{q}c_l^{|k_{\nu l}|/2} \quad \text{and} \quad \dot{b} = 2b^{s/2}. \tag{3.31}$$

We have that the invariant ray solution of the model system (3.29) exists, whether $c_j > 0$ for all $j = 1, \ldots, q$. Therefore, the system (3.29) has an invariant ray solution, if and only if, the conditions of the theorem are satisfied. In particular, the equilibrium solution $(0, 0)$ of the model Hamiltonian system (3.29) is unstable in the Liapunov sense. ■

Corollary 3.5. *Assume that the μ vectors of resonance vectors $k_1, \ldots, k_\mu$ do not have interaction, then all the components of each resonance vector must be positive and the condition* (3.26) *is reduced to*

$$c_j = |A_\nu|k_{\nu j}\prod_{l=1}^{q}c_l^{|k_{\nu l}|/2} > 0.$$

Proof. Suppose the existence of μ resonance vectors without interaction, say

$$\begin{array}{llllllllllll}
k_1 & = & (k_{11}, & \ldots, & k_{1q_1}, & 0, & \ldots, & 0, & 0, & \ldots, & 0, & 0, \ldots, 0) \\
 & \vdots & & & & & & & & & & \\
k_\mu & = & (0, & \ldots & 0, & 0, & \ldots & 0, & k_{\mu q_{\mu-1}+1}, & \ldots, & k_{\mu q}, & 0, \ldots, 0).
\end{array}$$

The Hamiltonian function in Lie normal form given in (3.22) assumes the simplified form

$$\begin{aligned} H &= H_2(I) + 2A_1\prod_{l=1}^{q_1}I_l^{|k_{1l}|/2}\cos(k_1\cdot\varphi) + 2A_2\prod_{l=q_1+1}^{q_2}I_l^{|k_{2l}|/2}\cos(k_2\cdot\varphi) + \ldots + \\ &\quad 2A_\mu\prod_{l=q_{\mu-1}+1}^{q}I_l^{|k_{\mu l}|/2}\cos(k_\mu\cdot\varphi), \end{aligned}$$

and its Hamiltonian system has the form

$$
\begin{aligned}
\dot{I}_j &= -2A_\nu k_{\nu j} \prod_{l=q_{\nu-1}+1}^{q_\nu} I_l^{k_{\nu l}/2} \sin(\theta_\nu), \quad j = q_{\nu-1}+1, \ldots, q_\nu, \\
\dot{I}_\alpha &= 0, \quad \alpha = q+1, \ldots, n, \\
\dot{\theta}_\nu &= -A_\nu \prod_{l=q_{\nu-1}+1}^{q_\nu} I_l^{|k_{\nu l}|/2} \cos(\theta_\nu) \sum_{j=q_{\nu-1}+1}^{q_\nu} \frac{|k_{\nu j}| k_{\nu j}}{I_j}, \quad \nu = 1, \ldots, \mu,
\end{aligned}
\tag{3.32}
$$

where $0 = q_0 < q_1 \leq \ldots \leq q_\mu = q$. Here, analogously to (3.30), we propose an unbounded and increasing invariant type solution for the Hamiltonian system model (3.32) such that

$$
I_j = c_j b(t),\ c_j > 0,\ j = 1, \ldots, q,\ \dot{b} = 2b^{s/2},\ \theta_\nu^0 = -\tfrac{\pi}{2}\tfrac{A_\nu}{|A_\nu|},\ \nu = 1, \ldots, \mu. \tag{3.33}
$$

But here we choose all the angles of the form $\pm\frac{\pi}{2}$. In fact, if such a solution exists, by replacing these expressions in the system (3.32), we arrive at

$$
\dot{I}_j = 2b(t)^{s/2} |A_\nu| k_{\nu j} \prod_{l=1}^{q} c_l^{|k_{\nu l}|/2}.
$$

As $I_j = c_j b(t)$, we take

$$
c_j = |A_\nu| k_{\nu j} \prod_{l=1}^{q} c_l^{|k_{\nu l}|/2} > 0 \quad \text{and} \quad \dot{b} = 2b^{s/2}.
$$

Thus, the proof is ended. ∎

Observation 3.6. *In particular, it follows by Corollary 3.5 that in the case of vectors of resonance without interaction a necessary condition for the existence of invariant ray solution is that every component of all resonance vectors must be positive.*

3.2.2 Proof of Theorem 3.4

We are going to prove the instability in the Liapunov sense of the origin of the complete Hamiltonian system, that is, where we introduce the perturbing terms in the Hamiltonian function model (3.23). The Hamiltonian systems in this case is

$$\begin{aligned}
\dot I_j &= -2\sum_{\nu=1}^{\mu} A_\nu k_{\nu j}\prod_{l=1}^{q} I_l^{k_{\nu l}/2}\sin(k_\nu\cdot\varphi)+\Gamma_j(I,\varphi), \quad j=1,\dots,q,\\
\dot I_\alpha &= \Gamma_\alpha(I,\varphi), \quad \alpha=q+1,\dots,n,\\
\dot\varphi_j &= -\omega_j-\sum_{\nu=1}^{\mu}\frac{1}{I_j}A_\nu|k_{\nu j}|\prod_{l=1}^{q} I_l^{|k_{\nu l}|/2}\cos(k_\nu\cdot\varphi)+\Theta_j(I,\varphi), \quad j=1,\dots,n.
\end{aligned} \tag{3.34}$$

where $\Gamma_j(I,\varphi)=\mathcal{O}\left(||I||^{(s+1)/2}\right), \Theta_j(I,\varphi)=\mathcal{O}\left(||I||^{(s-1)/2}\right)$, for $j=1,\dots,n$. Introducing the variables θ_ν as in the previous section, we arrive to

$$\begin{aligned}
\dot I_j &= 2\sum_{\nu=1}^{\mu} R_\nu Q_{\nu j}(\theta_\nu)+\Gamma_j(I,\varphi), \quad j=1,\dots,q\\
\dot I_\alpha &= \Gamma_\alpha(I,\varphi), \quad \alpha=q+1,\dots,n\\
\dot\theta_\nu &= \sum_{j=1}^{q}\sum_{\nu=1}^{\mu}\frac{1}{I_j}|k_{\nu j}|R_\nu Q'_{\nu j}(\theta_\nu)+\Theta_\nu(I,\varphi), \quad \nu=1,\dots,\mu.
\end{aligned} \tag{3.35}$$

To be able to control the higher order terms, we introduce the generalized $n-$ dimensional cylindrical coordinates $(\rho,\psi_1,\dots,\psi_{q-1},I_{q+1},\dots,I_n)$ defined by the formulas

$$\begin{aligned}
I_1 &= c_1\rho\cos\psi_1,\\
I_j &= c_j\rho\cos\psi_j\prod_{l=1}^{j-1}\sin\psi_l,\ j=2,\dots q-1,\\
I_q &= c_q\rho\prod_{l=1}^{q-1}\sin\psi_l,\\
I_\alpha &= I_\alpha,\ \alpha=q+1,\dots,n.
\end{aligned} \tag{3.36}$$

We will use the following lemmas which help us to prove our main result.

Lemma 3.7. *The Hamiltonian system* (3.35) *in the generalized cylindrical coordinates assumes the form*

$$\begin{aligned}
\dot\rho &= 2\rho^{s/2}\sum_{\nu=1}^{\mu} R_\nu^0\ F_1(\psi)\ F_2(\psi,\theta)+\mathcal{R}_0,\\
\dot I_\alpha &= \mathcal{R}_\alpha,\ \alpha=q+1,\dots,n,\\
\dot\psi_\beta &= 2\rho^{s/2-1}\sum_{\nu=1}^{\mu} R_\nu^0\ F_1(\psi)\ F_{3\beta}(\psi,\theta)+\mathcal{R}_\beta,\ \beta=1,\dots,q-1,\\
\dot\theta_\nu &= \rho^{s/2-1}\sum_{i=1}^{\mu} R_\nu^0\ F_1(\psi)\ F_4(\psi,\theta)+\mathcal{R}_\nu,\ \nu=1,\dots,\mu,
\end{aligned} \tag{3.37}$$

where $\mathcal{R}_0 \sim \mathcal{O}\left(||I_*||^{(s+1)/2}\right)$, $\mathcal{R}_\alpha \sim \mathcal{O}\left(||I_*||^{(s+1)/2}\right)$, $\mathcal{R}_\beta \sim \mathcal{O}\left(||I_*||^{(s-1)/2}\right)$, $\mathcal{R}_\nu \sim \mathcal{O}\left(||I_*||^{(s-1)/2}\right)$ *with* $I_* = (\rho, I_{q+1}, \ldots, I_n)$ *and*

$$\begin{aligned}
R_\nu^0 &= \prod_{l=1}^{q} c_l^{|k_{\nu l}|/2},\\
F_1(\psi) &= \prod_{l=1}^{q-1} \cos\psi_l^{|k_{\nu l}|/2} \prod_{l=j+1}^{q-1} \sin\psi_j^{|k_{\nu l}|/2},\\
F_2(\psi,\theta) &= \frac{Q_{\nu 1}(\theta_\nu)}{c_1}\cos\psi_1 + \sum_{j=2}^{q-1}\frac{Q_{\nu j}(\theta_\nu)}{c_j}\cos\psi_j\prod_{l=1}^{j-1}\sin\psi_l + \frac{Q_{\nu q}(\theta_\nu)}{c_q}\prod_{l=1}^{q-1}\sin\psi_l,\\
F_{31}(\psi,\theta) &= -\tfrac{Q_{\nu 1}(\theta_\nu)}{c_1}\sin\psi_1 + \cos\psi_1\ H_{31}(\psi,\theta),\\
F_{3\beta}(\psi,\theta) &= \frac{1}{\displaystyle\prod_{l=1}^{\beta-1}\sin\psi_l}\left[-\frac{Q_{\nu\beta}(\theta_\nu)}{c_\beta}\sin\psi_\beta + \cos\psi_\beta\ H_{3\beta}(\psi,\theta)\right],\\
H_{3\beta}(\psi,\theta) &= \frac{Q_{\nu,\beta+1}(\theta_\nu)}{c_{\beta+1}}\cos\psi_{\beta+1} + \frac{\displaystyle\sum_{j=\beta+2}^{q-1}\frac{Q_{\nu j}(\theta_\nu)}{c_j}\cos\psi_j\prod_{l=1}^{j-1}\sin\psi_l}{\displaystyle\prod_{l=1}^{\beta}\sin\psi_l} +\\
&\quad \frac{Q_{\nu q}(\theta_\nu)}{c_q}\prod_{l=\beta+1}^{q-1}\sin\psi_l,\\
H_{3,q-1}(\psi,\theta) &= \frac{Q_{\nu q}}{c_q},\\
F_4(\psi,\theta) &= \frac{|k_{\nu 1}|Q'_{\nu 1}(\theta_\nu)}{c_1\cos\psi_1} + \sum_{j=2}^{q-1}\frac{|k_{\nu j}|Q'_{\nu j}(\theta_\nu)}{c_j\cos\psi_j\displaystyle\prod_{l=1}^{j-1}\sin\psi_l} + \frac{|k_{\nu q}|Q'_{\nu q}(\theta_\nu)}{c_q\displaystyle\prod_{l=1}^{q-1}\sin\psi_l}
\end{aligned} \tag{3.38}$$

with $\beta = 2, \ldots, q-2$.

Proof. Easily, it follows that $\rho^2 = \displaystyle\sum_{j=1}^{n}\frac{I_j^2}{c_j^2}$, then differentiating in (3.36) we arrive at

$$\begin{aligned}
\dot\rho &= 2\rho^{s/2}\sum_{\nu=1}^{\mu}\prod_{l=1}^{q} c_l^{|k_{\nu l}|/2}\prod_{j=1}^{q-1}\cos\psi_j^{|k_{\nu j}|/2}\prod_{l=j+1}^{q-1}\sin\psi_j^{|k_{\nu l}|/2}\times\\
&\left(\frac{Q_{\nu 1}(\theta_\nu)}{c_1}\cos\psi_1 + \sum_{j=2}^{q-1}\frac{Q_{\nu j}(\theta_\nu)}{c_j}\cos\psi_j\prod_{l=1}^{j-1}\sin\psi_l + \prod_{l=1}^{q-1}\frac{Q_{\nu q}(\theta_\nu)}{c_q}\sin\psi_l\right) +\\
&\mathcal{O}\left(||I_*||^{(s+1)/2}\right).
\end{aligned}$$

For $\beta = 1, \ldots, q-2$ by induction we are going to prove that

$$\begin{aligned}\dot{\psi}_\beta &= 2\rho^{s/2-1} \sum_{\nu=1}^{\mu} \prod_{l=1}^{q} c_l^{|k_{\nu l}|/2} \prod_{j=1}^{q-1} \cos\psi_j^{|k_{\nu j}|/2} \prod_{l=j+1}^{q-1} \sin\psi_j^{|k_{\nu l}|/2} \times \\ &\frac{1}{\prod_{l=1}^{\beta-1}\sin\psi_l}\left[-\frac{Q_{\nu\beta}}{c_\beta}\sin\psi_\beta + \cos\psi_\beta\left(\frac{Q_{\nu,\beta+1}}{c_{\beta+1}}\cos\psi_{\beta+1} + \frac{\sum_{j=\beta+2}^{q-1}\frac{Q_{\nu j}}{c_j}\cos\psi_j\prod_{l=1}^{j-1}\sin\psi_l}{\prod_{l=1}^{\beta}\sin\psi_l} + \right.\right. \\ &\left.\left.\frac{Q_{\nu q}}{c_q}\prod_{l=\beta+1}^{q-1}\sin\psi_l\right)\right] + \mathcal{O}\left(||I_*||^{(s-1)/2}\right).\end{aligned} \tag{3.39}$$

In fact, for $\beta = 1$ by (3.36) we have $I_1 = c_1\rho\cos\psi_1$, then

$$\begin{aligned}\dot{\psi}_1 &= 2\rho^{s/2-1} \sum_{\nu=1}^{\mu} \prod_{l=1}^{q} c_l^{|k_{\nu l}|/2} \prod_{j=1}^{q-1} \cos\psi_j^{|k_{\nu j}|/2} \prod_{l=j+1}^{q-1} \sin\psi_j^{|k_{\nu l}|/2} \times \\ &\left[-\tfrac{Q_{\nu 1}}{c_1}\sin\psi_1 + \cos\psi_1\left(\tfrac{Q_{\nu 2}}{c_2}\cos\psi_2 + \frac{\sum_{j=3}^{q-1}\frac{Q_{\nu j}}{c_j}\cos\psi_j\prod_{l=1}^{j-1}\sin\psi_l}{\sin\psi_1} + \tfrac{Q_{\nu q}}{c_q}\prod_{l=2}^{q-1}\sin\psi_l\right)\right] + \\ &\mathcal{O}\left(||I_*||^{(s-1)/2}\right).\end{aligned}$$

Next, supposing that (3.39) is true for β, then from (3.36) we have the relations $\frac{I_{\beta+1}}{I_\beta} = \frac{c_{\beta+1}}{c_\beta}\cos\psi_{\beta+1}\tan\psi_\beta$, so we get

$$\begin{aligned}\dot{\psi}_{\beta+1} &= 2\rho^{s/2-1} \sum_{\nu=1}^{\mu} \prod_{l=1}^{q} c_l^{|k_{\nu l}|/2} \prod_{j=1}^{q-1} \cos\psi_j^{|k_{\nu j}|/2} \prod_{l=j+1}^{q-1} \sin\psi_j^{|k_{\nu l}|/2} \times \\ &\frac{1}{\prod_{l=1}^{\beta}\sin\psi_l}\left[-\frac{Q_{\nu,\beta+1}}{c_{\beta+1}}\sin\psi_{\beta+1} + \cos\psi_{\beta+1}\left(\frac{Q_{\nu,\beta+2}}{c_{\beta+2}}\cos\psi_{\beta+2} + \right.\right. \\ &\left.\left.\frac{\sum_{j=\beta+3}^{q-1}\frac{Q_{\nu j}}{c_j}\cos\psi_j\prod_{l=1}^{j-1}\sin\psi_l}{\prod_{l=1}^{\beta+1}\sin\psi_l} + \frac{Q_{\nu q}}{c_q}\prod_{l=\beta+2}^{q-1}\sin\psi_l\right)\right] \\ &+\mathcal{O}\left(||I_*||^{(s-1)/2}\right).\end{aligned}$$

Thus (3.39) holds for $\beta+1$, and the proof of the induction steep is complete. Now, we will determine $\dot{\psi}_{q-1}$ from (3.36), note that by the relations $\frac{I_q}{I_{q-1}} = \frac{c_q}{c_{q-1}} \tan\psi_{q-1}$, we get

$$\begin{aligned}\dot{\psi}_{q-1} &= 2\rho^{s/2-1}\sum_{\nu=1}^{\mu}\prod_{l=1}^{q} c_l^{|k_{\nu l}|/2}\prod_{j=1}^{q-1}\cos\psi_j^{|k_{\nu j}|/2}\prod_{l=j+1}^{q-1}\sin\psi_j^{|k_{\nu l}|/2}\times\\ &\frac{1}{\prod_{l=1}^{q-2}\sin\psi_l}\left(-\frac{Q_{\nu,q-1}}{c_{q-1}}\sin\psi_{q-1}+\frac{Q_{\nu q}}{c_q}\cos\psi_{q-1}\right)+\mathcal{O}\left(||I_*||^{(s-1)/2}\right).\end{aligned}$$

Substituting (3.36) in (3.28) we arrive at

$$\begin{aligned}\dot{\theta}_\nu &= \rho^{s/2-1}\sum_{i=1}^{\mu}\prod_{l=1}^{q} c_l^{|k_{\nu l}|/2}\prod_{j=1}^{q-1}\cos\psi_j^{|k_{\nu j}|/2}\prod_{l=j+1}^{q-1}\sin\psi_j^{|k_{\nu l}|/2}\times\\ &\left(\frac{|k_{\nu 1}|Q'_{\nu 1}(\theta_\nu)}{c_1\cos\psi_1}+\sum_{j=2}^{q-1}\frac{|k_{\nu j}|Q'_{\nu j}(\theta_\nu)}{c_j\cos\psi_j\prod_{l=1}^{j-1}\sin\psi_l}+\frac{|k_{\nu q}|Q'_{\nu q}(\theta_\nu)}{c_q\prod_{l=1}^{q-1}\sin\psi_l}\right)+\mathcal{O}\left(||I_*||^{(s-1)/2}\right).\end{aligned}$$

■

From now on, we will call as truncated angular system, the Hamiltonian system associated to $\dot{\psi}_\beta$, $\beta = 1,\ldots,q-1$ and $\dot{\theta}_\nu$, $\nu = 1,\ldots,\mu$ in the system (3.37), eliminating the higher order terms. Taking the previous coordinates and using the previous lemma we have that the invariant ray solution satisfies the following property.

Lemma 3.8. *The invariant ray solution* (3.30) *of the model Hamiltonian system given in* (3.29) *in the new coordinates corresponds to the point*

$$\psi = \psi_\beta^0, \quad \cos\psi_\beta^0 = \frac{1}{\sqrt{q-\beta+1}}, \quad \sin\psi_\beta^0 = \frac{\sqrt{q-\beta}}{\sqrt{q-\beta+1}}, \quad \theta = \theta_\nu^0, \quad b(t) = \frac{1}{\sqrt{q}}\rho(t), \tag{3.40}$$

for $\beta = 1,\ldots,q-1$ *and* $\nu = 1,\ldots,\mu$. *Furthermore, it is verified that* (3.40) *is an equilibrium solution of the truncated angular Hamiltonian system.*

Proof. In fact, by virtue of the form of the invariant ray solution in action-angle variables (3.30), we have $I_j/c_j = b(t)$, for all $j = 1,\ldots,q$. In particular, $I_j/c_j = I_q/c_q$, that is, by (3.36) we must have $\cos\psi_j = \prod_{l=j}^{q-1}\sin\psi_l$, from which is obtained $\tan\psi_{q-1} = 1$, $\tan\psi_{q-2} = \sqrt{2}, \tan\psi_{q-3} = \sqrt{3},\ldots,\tan\psi_{q-j} = \sqrt{j}$. Thus,

$\sin\psi_{q-j} = \sqrt{j}/\sqrt{j+1}$, $\cos\psi_{q-j} = 1/\sqrt{j+1}$. Doing $q-j=\beta$, we obtain that $\cos\psi_\beta^0 = (q-\beta+1)^{-1/2}$, $\sin\psi_\beta^0 = \sqrt{q-\beta}/\sqrt{q-\beta+1}$. Then, from $I_1 = c_1\rho\cos\psi_1$ and $I_1 = c_1 b(t)$, it follows that $b(t) = \rho\cos\psi_1^0 = \rho/\sqrt{q}$ and therefore the invariant ray solution (3.30) in the new coordinates (3.36) assumes the form (3.40). ■

Next, we move the equilibrium solution (ψ^0, θ^0) to the origin by means of the following change of coordinates $\psi_\beta = \overline{\psi}_\beta + \psi_\beta^0$, $\beta = 1, \ldots, q-1$, $\theta_\nu = \overline{\theta}_\nu + \theta_\nu^0$, $\nu = 1, \ldots, \mu$, so that the Hamiltonian system (3.37) assumes the form

$$\begin{aligned}
\dot{\rho} &= 2\rho^{s/2}\sum_{\nu=1}^{\mu} R_\nu^0\ F_1(\overline{\psi}+\psi^0)\ F_2(\overline{\psi}+\psi^0, \overline{\theta}+\theta^0) + \mathcal{R}_0,\\
\dot{I}_\alpha &= \mathcal{R}_\alpha,\ \alpha = q+1, \ldots, n,\\
\dot{\overline{\psi}}_\beta &= 2\rho^{s/2-1}\sum_{\nu=1}^{\mu} R_\nu^0\ F_1(\overline{\psi}+\psi^0)\ F_{3\beta}(\overline{\psi}+\psi^0, \overline{\theta}+\theta^0) + \mathcal{R}_\beta,\ \beta = 1, \ldots, q-1,\\
\dot{\overline{\theta}}_\nu &= \rho^{s/2-1}\sum_{i=1}^{\mu} R_\nu^0\ F_1(\overline{\psi}+\psi^0)\ F_4(\overline{\psi}+\psi^0, \overline{\theta}+\theta^0) + \mathcal{R}_\nu,\ \nu = 1, \ldots, \mu.
\end{aligned} \tag{3.41}$$

In the following result we characterize the system (3.41) expanded in Taylor series around the point $(\overline{\psi}^0, \overline{\theta}^0) = (0,0)$.

Lemma 3.9. *The Hamiltonian system* (3.37)*, expanded in Taylor series in a neighborhood of the point* $(\overline{\psi}^0, \overline{\theta}^0) = (0,0)$ *assumes the form*

$$\begin{aligned}
\dot{\rho} &= 2\rho^{s/2}q^{(2-s)/4} + \overline{\mathcal{R}}_0, \quad \dot{I}_\alpha = \overline{\mathcal{R}}_\alpha,\ \alpha = q+1, \ldots, n,\\
\dot{\overline{\psi}}_\beta &= \rho^{s/2-1}q^{(2-s)/4}\sum_{h=1}^{q-1} B_{\beta h}\overline{\psi}_h + \overline{\mathcal{R}}_\beta,\ \beta = 1, \ldots, q-1\\
\dot{\overline{\theta}}_\nu &= \rho^{s/2-1}q^{(2-s)/4}\sum_{i=1}^{\mu} B_{n+\nu,n+i}\overline{\theta}_i + \overline{\mathcal{R}}_{n+\nu},\ \nu = 1, \ldots, \mu,
\end{aligned} \tag{3.42}$$

where $\overline{\psi}_* = (\overline{\psi}_1, \ldots, \overline{\psi}_{q-1})$, $\overline{\theta}_* = (\overline{\theta}_1, \ldots, \overline{\theta}_\mu)$, $B_{n+\nu,n+i} = -R_i^0 \sum_{j=1}^{q} \frac{|k_{\nu j}|Q_{ij}^0}{c_j}$,

$$\begin{aligned}
B_{\beta h} = &\sum_{\nu=1}^{\mu} \frac{2R_\nu^0}{(q-\beta+1)\sqrt{q-\beta}}\left(\sum_{j=\beta+1}^{q}\frac{Q_{\nu j}^0}{c_j} - \frac{Q_{\nu\beta}^0}{c_\beta}(q-\beta)\right)\frac{1}{2\sqrt{q-h}}\times\\
&\left(\sum_{l=h+1}^{q}|k_{\nu l}| - |k_{\nu h}|(q-h)\right) - 2\delta_{\beta h},
\end{aligned}$$

and the functions $\overline{\mathcal{R}}$ have the following structure

$$
\begin{aligned}
\overline{\mathcal{R}}_0(I_*,\overline{\psi}_*,\overline{\theta}_*) &= \overline{\mathcal{R}}_0^{(1)}(I_*,\overline{\psi}_*,\overline{\theta}_*) + \rho^{s/2}\overline{\mathcal{R}}_0^{(2)}(I_*,\overline{\psi}_*,\overline{\theta}_*),\\
\overline{\mathcal{R}}_\alpha(I_*,\overline{\psi}_*,\overline{\theta}_*) &\sim \mathcal{O}\left(||I_*||^{(s+1)/2}\right),\\
\overline{\mathcal{R}}_\beta(I_*,\overline{\psi}_*,\overline{\theta}_*) &= \overline{\mathcal{R}}_\beta^{(1)}(I_*,\overline{\psi}_*,\overline{\theta}_*) + \rho^{s/2-1}\overline{\mathcal{R}}_\beta^{(2)}(I_*,\overline{\psi}_*,\overline{\theta}_*),\\
\overline{\mathcal{R}}_{n+\nu}(I_*,\overline{\psi}_*,\overline{\theta}_*) &= \overline{\mathcal{R}}_{n+\nu}^{(1)}(I_*,\overline{\psi}_*,\overline{\theta}_*) + \rho^{s/2-1}\overline{\mathcal{R}}_{n+\nu}^{(2)}(I_*,\overline{\psi}_*,\overline{\theta}_*),
\end{aligned}
$$

with

$$
\begin{aligned}
\overline{\mathcal{R}}_0^{(1)} &\sim \mathcal{O}\left(||I_*||^{(s+1)/2}\right), \quad \overline{\mathcal{R}}_0^{(2)}(0,\overline{\psi}_*,\overline{\theta}_*) \sim \mathcal{O}\left(||(\overline{\psi}_*,\overline{\theta}_*)||\right),\\
\overline{\mathcal{R}}_\beta^{(1)} &\sim \mathcal{O}\left(||I_*||^{(s-1)/2}\right), \quad \overline{\mathcal{R}}_\beta^{(2)}(0,\overline{\psi}_*,\overline{\theta}_*) \sim \mathcal{O}\left(||(\overline{\psi}_*,\overline{\theta}_*)||^2\right),\\
\overline{\mathcal{R}}_{n+\nu}^{(1)} &\sim \mathcal{O}\left(||I_*||^{(s-1)/2}\right), \quad \overline{\mathcal{R}}_{n+\nu}^{(2)}(0,\overline{\psi}_*,\quad\overline{\theta}_*) \sim \mathcal{O}\left(||(\overline{\psi}_*,\overline{\theta}_*)||^2\right).
\end{aligned}
$$

Proof. First, we note that

$$
\begin{aligned}
\tfrac{\partial F_{3\beta}}{\partial\theta_\nu}(\psi^0,\theta^0) &= 0,\\
\tfrac{\partial F_{3\beta}}{\partial\psi_h}(\psi^0,\theta^0) &= \tfrac{q^{1/2}}{(q-\beta+1)\sqrt{q-\beta}\sqrt{q-h}}\left[\sum_{j=h+1}^{q}\frac{Q_{\nu j}^0}{c_j} - \frac{Q_{\nu h}^0}{c_h}(q-h)\right],\\
F_1(\psi^0) &= q^{-s/4}, \quad F_2(\psi^0,\theta^0) = q^{-1/2}\sum_{j=1}^{q}\frac{Q_{\nu j}^0}{c_j}, \quad F_4(\psi^0,\theta^0)=0,\\
\frac{\partial F_4}{\partial\theta_i}(\psi^0,\theta^0) &= -q^{1/2}\sum_{j=1}^{q}\frac{|k_\nu j|Q_{ij}^0}{c_j},\\
\tfrac{\partial F_2}{\partial\theta_\nu}(\psi^0,\theta^0) &= 0, \quad \tfrac{\partial F_4}{\partial\psi_h}(\psi^0,\theta^0)=0,\\
F_{3\beta}(\psi^0,\theta^0) &= \tfrac{q^{1/2}}{(q-\beta+1)\sqrt{q-\beta}}\left(\sum_{j=\beta+1}^{q}\frac{Q_{\nu j}^0}{c_j} - \frac{Q_{\nu\beta}^0}{c_\beta}(q-\beta)\right),\\
\tfrac{\partial F_2}{\partial\psi_h}(\psi^0,\theta^0) &= \tfrac{q^{-1/2}}{\sqrt{q-h}}\left(\sum_{j=h+1}^{q}\frac{Q_{\nu j}^0}{c_j} - \frac{Q_{\nu h}^0}{c_h}(q-h)\right),\\
\tfrac{\partial F_1}{\partial\psi_h}(\psi^0,\theta^0) &= \tfrac{q^{-s/2}}{2\sqrt{q-h}}\left(\sum_{l=h+1}^{q}|k_{\nu l}| - |k_{\nu h}|(q-h)\right),
\end{aligned}
$$

with $Q_{\nu j}^0 = Q_{\nu j}(\theta_\nu^0)$. Now, expanding system (3.41) in Taylor series in a neighborhood around the point $(\overline{\psi}^0,\overline{\theta}^0) = (0,0)$ and using the notation $(\overline{\psi}^0,\overline{\theta}^0) =$

$(\overline{\psi}_1^0, \ldots, \overline{\psi}_{q-1}^0, \overline{\theta}_1^0, \ldots, \overline{\theta}_\mu^0)$ and $(\psi^0, \theta^0) = (\psi_1^0, \ldots, \psi_{q-1}^0, \theta_1^0, \ldots, \theta_\nu^0)$, we arrive at

$$
\begin{aligned}
\dot{\rho} &= 2\rho^{s/2} q^{(2-s)/4} + \mathcal{O}\left(||I_*||^{(s+1)/2}\right) + \rho^{s/2}\mathcal{O}\left(||(\overline{\psi}_*, \overline{\theta}_*)||\right), \\
\dot{I}_\alpha &= \mathcal{O}\left(||I_*||^{(s+1)/2}\right), \\
\dot{\overline{\psi}}_\beta &= \rho^{s/2-1} q^{(2-s)/4} \sum_{h=1}^{q-1} \left[\sum_{\nu=1}^{\mu} \frac{2R_\nu^0}{(q-\beta+1)\sqrt{q-\beta}} \left(\sum_{j=\beta+1}^{q} \frac{Q_{\nu j}^0}{c_j} - \frac{Q_{\nu\beta}^0}{c_\beta}(q-\beta) \right) \times \right. \\
& \left. \frac{1}{2\sqrt{q-h}} \left(\sum_{l=h+1}^{q} |k_{\nu l}| - |k_{\nu h}|(q-h) \right) - 2\delta_{\beta h} \right] \overline{\psi}_h + \mathcal{O}\left(||I_*||^{(s-1)/2}\right) + \\
& \rho^{s/2-1}\mathcal{O}\left(||(\overline{\psi}_*, \overline{\theta}_*)||^2\right), \\
\dot{\overline{\theta}}_\nu &= \rho^{s/2-1} q^{(2-s)/4} \sum_{i=1}^{\mu} \left(-R_i^0 \sum_{j=1}^{q} \frac{|k_{\nu j}| Q_{ij}^0}{c_j} \right) \overline{\theta}_i + \mathcal{O}\left(||I_*||^{(s-1)/2}\right) + \\
& \rho^{s/2-1}\mathcal{O}\left(||(\overline{\psi}_*, \overline{\theta}_*)||^2\right).
\end{aligned}
$$

■

Next, with the purpose that the terms of higher order in (3.42) evaluated at $\overline{\psi} = 0, \overline{\theta} = 0$ depend only on I_*, we introduce the convenient transformation $(\overline{\psi}, \overline{\theta}) \to (\psi^*, \theta^*)$

$$
\begin{aligned}
\overline{\psi}_\beta &:= \psi_\beta^* + \Psi_\beta = \psi_\beta^* + \sum_{l=1}^{2(1+\gamma)} c_{\beta l}\rho^{l/2}, \quad \beta = 1, \ldots, q-1, \\
\overline{\theta}_\nu &:= \theta_\nu^* + \Theta_\nu = \theta_\nu^* + \sum_{l=1}^{2(1+\gamma)} d_{\nu l}\rho^{l/2}, \quad \nu = 1, \ldots \mu,
\end{aligned} \tag{3.43}
$$

for some adequate natural number γ and real coefficients $c_{\beta l}$ and $d_{\nu l}$.

Lemma 3.10. *Under the condition* $\det(B_\nu\zeta - NI) \neq 0$*, for* $N = 1, \ldots, 2(1+\gamma)$*, it is possible to construct the transformation in* (3.43) *such that the Hamiltonian system* (3.42) *in the coordinates* (ψ^*, θ^*) *assumes the form*

$$
\begin{aligned}
\dot{\rho} &= 2\varkappa\rho^{s/2} + \mathcal{R}_0^*, \\
\dot{I}_\alpha &= \mathcal{R}_\alpha^*, \ \alpha = q+1, \ldots, n, \\
\dot{\psi}_\beta^* &= \varkappa\rho^{s/2-1} \sum_{h=1}^{q-1} B_{\beta h}\psi_h^* + \mathcal{R}_\beta^*, \ \beta = 1, \ldots, q-1, \\
\dot{\theta}_\nu^* &= \varkappa\rho^{s/2-1} \sum_{i=1}^{\mu} B_{n+\nu, n+i}\theta_i^* + \mathcal{R}_{n+\nu}^*, \ \nu = 1, \ldots, \mu,
\end{aligned} \tag{3.44}
$$

where $\varkappa = q^{(2-s)/4}$, $\psi^* = (\psi_1^*, \ldots, \psi_{q-1}^*)$ *and* $\theta^* = (\theta_1^*, \ldots, \theta_\mu^*)$ *and the functions* $\mathcal{R}^*$ *have the following structure*

$$\begin{aligned}
\mathcal{R}_0^*(I_*, \psi^*, \theta^*) &= \mathcal{R}_0^{(1)}(I_*, \psi^*, \theta^*) + \rho^{s/2}\mathcal{R}_0^{(2)}(I_*, \psi^*, \theta^*),\\
\mathcal{R}_\alpha^*(I_*, \psi^*, \theta^*) &\sim \mathcal{O}\left(||I_*||^{(s+1)/2}\right),\\
\mathcal{R}_\beta^*(I_*, \psi^*, \theta^*) &= \mathcal{R}_\beta^{(1)}(I_*, \psi^*, \theta^*) + \rho^{s/2-1}\mathcal{R}_\beta^{(2)}(I_*, \psi^*, \theta^*),\\
\mathcal{R}_{n+\nu}^*(I_*, \psi^*, \theta^*) &= \mathcal{R}_{n+\nu}^{(1)}(I_*, \psi^*, \theta^*) + \rho^{s/2-1}\mathcal{R}_{n+\nu}^{(2)}(I_*, \psi^*, \theta^*),
\end{aligned} \tag{3.45}$$

with

$$\begin{aligned}
\mathcal{R}_0^{(1)} &\sim \mathcal{O}\left(||I_*||^{(s+1)/2}\right), \quad \mathcal{R}_0^{(2)}(0, \psi^*, \theta^*) \sim \mathcal{O}\left(||(\psi^*, \theta^*)||\right),\\
\mathcal{R}_\beta^{(1)} &\sim \mathcal{O}\left(||I_*||^{(s-1)/2}\right), \quad \mathcal{R}_\beta^{(2)}(0, \psi^*, \theta^*) \sim \mathcal{O}\left(||(\psi^*, \theta^*)||^2\right),\\
\mathcal{R}_{n+\nu}^{(1)} &\sim \mathcal{O}\left(||I_*||^{(s-1)/2}\right), \quad \mathcal{R}_{n+\nu}^{(2)}(0, \psi^*, \theta^*) \sim \mathcal{O}\left(||(\psi^*, \theta^*)||^2\right).
\end{aligned}$$

such that the rest functions $\mathcal{R}^* = \mathcal{R}^*(I_*, \psi^*, \theta^*)$ *valued at* $\psi^* = 0, \theta^* = 0$ *depend only on* I_**, that is,*

$$\mathcal{R}_0^*(I_*, 0, 0) = \mathcal{R}_0^*(I_*),\ \mathcal{R}_\beta^*(I_*, 0, 0) = \mathcal{R}_\beta^*(I_*),\ \mathcal{R}_{n+\nu}^*(I_*, 0, 0) = \mathcal{R}_{n+\nu}^*(I_*).$$

Proof. We make a convenient change of coordinate in ψ^*, θ^*, as a perturbation of the identity, such that perturbing terms are given by a sum of powers of ρ to a convenient order. This order is chosen for that the remainders $\overline{\mathcal{R}}$ in (3.42) evaluated in $\psi^* = 0, \theta^* = 0$ should depend only of I_* and to construct the Chetaev's function. Since the transformation (3.43) is a perturbation of identity, the Hamiltonian system (3.42) assume the form (3.44). Now, we look for conditions on the coefficients $B_{\beta h}$ and $B_{n+\nu,n+i}$ in (3.44) for the existence of $c_{\beta l}$ and $d_{\nu l}$ of the transformation (3.43). In fact, we differentiate the relations (3.43) with respect to t, we obtain

$$\begin{aligned}
\dot{\overline{\psi}}_\beta &= \dot{\psi}_\beta^* + \sum_{l=1}^{2(1+\gamma)} \frac{l}{2} c_{\beta l}\rho^{l/2-1}\dot{\rho},\ \beta = 1, \ldots, q-1,\\
\dot{\overline{\theta}}_\nu &= \dot{\theta}_\nu^* + \sum_{l=1}^{2(1+\gamma)} \frac{l}{2} d_{\nu l}\rho^{l/2-1}\dot{\rho},\ \nu = 1, \ldots, \mu.
\end{aligned} \tag{3.46}$$

From where we arrived

$$\begin{aligned}
\dot{\psi}_\beta^* &= \varkappa\rho^{s/2-1}\left(\sum_{h=1}^{q-1} B_{\beta h}\left(\psi_h^* + \Psi_h\right) - \sum_{l=1}^{2(1+\gamma)} l\ c_{\beta l}\rho^{l/2}\right) + \ldots,\\
\dot{\theta}_\nu^* &= \varkappa\rho^{s/2-1}\left(\sum_{i=1}^{\mu} B_{n+\nu,n+i}\left(\theta_\nu^* + \Theta_\nu\right) - \sum_{l=1}^{2(1+\gamma)} l\ d_{\nu l}\rho^{l/2}\right) + \ldots.
\end{aligned} \tag{3.47}$$

Doing $\psi_h^* = \theta_i^* = 0$ for $h = 1, \dots, q-1$ and $i = 1, \dots, \mu$ in (3.47) and developing the right hand side in each case we obtain

$$\begin{aligned}
\rho^{s/2-1} \sum_{l=1}^{2(1+\gamma)} a_{\beta l} \rho^{l/2} &= \varkappa \rho^{s/2-1} \left(\sum_{h=1}^{q-1} B_{\beta h} \sum_{l=1}^{2(1+\gamma)} c_{\beta l} \rho^{l/2} - \sum_{l=1}^{2(1+\gamma)} l \, c_{\beta l} \rho^{l/2} \right) \\
&= \varkappa \rho^{s/2-\frac{1}{2}} \left(\sum_{h=1}^{q-1} B_{\beta h} c_{\beta 1} - c_{\beta 1} \right) + \\
&\varkappa \rho^{s/2} \left(\sum_{h=1}^{q-1} B_{\beta h} c_{\beta 2} - 2 c_{\beta 2} \right) + \dots \\
\rho^{s/2-1} \sum_{l=1}^{2(1+\gamma)} b_{\nu l} \rho^{l/2} &= \varkappa \rho^{s/2-1} \left(\sum_{i=1}^{\mu} B_{n+\nu, n+i} \sum_{l=1}^{2(1+\gamma)} d_{\nu l} \rho^{l/2} - \sum_{l=1}^{2(1+\gamma)} l \, d_{\nu l} \rho^{l/2} \right) \\
&= \varkappa \rho^{s/2-\frac{1}{2}} \left(\sum_{i=1}^{\mu} B_{n+\nu, n+i} d_{\nu 1} - d_{\nu 1} \right) + \\
&\varkappa \rho^{s/2} \left(\sum_{i=1}^{\mu} B_{n+\nu, n+i} d_{\nu 2} - 2 d_{\nu 2} \right) + \dots
\end{aligned} \tag{3.48}$$

By comparison of the powers of ρ in equations (3.48), we obtain the system of equations

$$(B_{\nu\zeta} - NI) f_N = g_N, \tag{3.49}$$

where $f_N = (c_{1N}, \dots, c_{q-1,N}, d_{1N}, \dots, d_{\mu,N})$, $g_N = (a_{1N}, \dots, a_{q-1,N}, b_{1N}, \dots, b_{\mu,N})$, I is identity matrix and the matrix $B_{\nu\zeta}$ is as follows

$$B_{\nu\zeta} = \left(\begin{array}{ccc|ccc}
B_{11} & \dots & B_{1,q-1} & & & \\
\vdots & \ddots & \vdots & & \mathbf{0} & \\
B_{q-1,1} & \dots & B_{q-1,q-1} & & & \\
\hline
& & & B_{n+1,n+1} & \dots & B_{n+1,n+\mu} \\
& \mathbf{0} & & \vdots & \ddots & \vdots \\
& & & B_{n+\mu,n+1} & \dots & B_{n+\mu,n+\mu}
\end{array} \right). \tag{3.50}$$

Then the system (3.49) has a unique solution, provided that $\det(B_{\nu\zeta} - NI) \neq 0$, $N = 1, 2, \dots, 2(1+\gamma)$. Therefore, as a result of the transformation (3.43) and under the choice of the coefficients $c_{\beta l}$ and $d_{\nu l}$ we obtain

$$\mathcal{R}_0^*(I_*, 0, 0) = \mathcal{R}_0^*(I_*), \ \mathcal{R}_\beta^*(I_*, 0, 0) = \mathcal{R}_\beta^*(I_*), \ \mathcal{R}_{n+\nu}^*(I_*, 0, 0) = \mathcal{R}_{n+\nu}^*(I_*)$$

and $\mathcal{R}_\nu(\rho, \psi^* = 0, \theta^* = 0) = \mathcal{O}\left(\rho^{(s+1)/2+\gamma}\right)$ with $\nu = 1, \ldots, n+\mu$ but $\nu \neq q, \ldots, n$. This concludes the proof of the lemma. ■

Proof. Theorem 3.4. In order to analyze the instability of the origin associated the full Hamiltonian system (3.35), which is equivalent to study the instability of the origin of the Hamiltonian system (3.44), we are going to built a proper Chetaev's function.

Initially we consider the function $V = \rho$, and looking for $\dot{\rho}$ in (3.44), we see that there is a neighborhood of the growing solution in the form of an invariant ray (associated with the Hamiltonian system model (3.29)) where $V\dot{V} > 0$ for each $0 < ||I_*|| < \tau$ with τ small enough, which defines the cone K_1. Then we introduce the auxiliary functions

$$W_\beta = \left(\psi_\beta^*\right)^2 - \rho^{2(1+\gamma)},\ W_{n+\nu} = (\theta_\nu^*)^2 - \rho^{2(1+\gamma)},\ W_\alpha = I_\alpha^2 - \rho^{2(1+\gamma)},$$

with $\beta = 1, \ldots, q-1, \nu = 1, \ldots, \mu, \alpha = q+1, \ldots, n$. With these functions we define a second cone

$$K_2 = \max_i \{W_i \leq 0, \quad i = 1, \ldots n+\mu, \quad i \neq q\}.$$

Since the invariant ray type solution is contained in the cone K_1 and since we will choice γ properly later, $K_2 \subset K_1$ with $0 < ||I_*|| < \tau$.

Consider the region $\Omega = \{V > 0\} \cap K_1$. Note that $\Omega \neq \emptyset$ because the invariant ray is contained in the cone K_1. In addition to the fact that the origin is at the border of Ω, since $r = 0$ implies $\rho = 0$ and $I_\alpha = 0$ for $\alpha = q+1, \ldots, n$. Then $V = 0$, on the boundary of Ω given that $\theta_\nu^* = \psi_\beta^* = I_\alpha = \rho^{1+\gamma} = 0$ for $\nu = 1, \ldots, \mu$, $\alpha = q+1, \ldots, n$, $\beta = 1, \ldots, q-1$.

We compute that on the surface K_2 we must have $\psi_\beta^* = f_\beta(\psi_\beta^*, \rho)\rho^{(1+\gamma)}, \theta_\nu^* = f_\nu(\theta_\nu^*, \rho)\rho^{(1+\gamma)}$, and $I_\alpha = f_\alpha(I_\alpha, \rho)\rho^{(1+\gamma)}$, for certain differentiable functions f_β, f_ν and f_α such that $|f_\beta| < 1, |f_\nu| < 1, 0 < f_\alpha < 1$. On the other hand, on the lateral surfaces of the cone K_2, that is, $W_\beta = 0,\ W_{n+\nu} = 0,\ W_\alpha = 0$ we obtain

$$\begin{array}{llllllll}
\psi_\beta & = \pm\rho^{1+\gamma}, & \theta_\nu & = f_\nu\rho^{1+\gamma}, & I_\alpha & = f_\alpha\rho^{1+\gamma}, & |f_\beta| & \leq 1 \\
\psi_\beta & = f_\beta\rho^{1+\gamma}, & \theta_\nu & = \pm\rho^{1+\gamma}, & I_\alpha & = f_\alpha\rho^{1+\gamma}, & |f_\nu| & \leq 1 \\
\psi_\beta & = f_\beta\rho^{1+\gamma}, & \theta_\nu & = f_\nu\rho^{1+\gamma}, & I_\alpha & = \rho^{1+\gamma}, & |f_\alpha| & \leq 1,
\end{array}$$

respectively.

Now, we calculate $\dot{W}_i$ and estimate the sign of the derivative on the surface of the cone K_2 (which is contained in the cone K_1) and denote these values of derivatives as $\dot{W}_i^0$. Differentiating the functions W_i with $i = 1, \dots n+\mu$, $i \neq q$ we get

$$\begin{aligned}
\dot{W}_\beta &= 2\psi_\beta^* \dot{\psi}_\beta^* - 2(1+\gamma)\rho^{1+2\gamma}\dot{\rho}, \\
\dot{W}_{n+\nu} &= 2\theta_\nu^* \dot{\theta}_\nu^* - 2(1+\gamma)\rho^{1+2\gamma}\dot{\rho}, \\
\dot{W}_\alpha &= 2I_\alpha \dot{I}_\alpha - 2(1+\gamma)\rho^{1+2\gamma}\dot{\rho},
\end{aligned} \tag{3.51}$$

with $\beta = 1, \dots, q-1$, $\nu = 1, \dots, \mu$, $\alpha = q+1, \dots, n$. Next, we remark that the Hamiltonian system (3.44) on the lateral surfaces of the cone K_2 assumes the form

$$\begin{aligned}
\dot{\rho} &= 2\varkappa\rho^{s/2} + \mathcal{O}\left(\rho^{s/2+2\gamma+1}\right), \\
\dot{I}_\alpha &= \mathcal{O}\left(\rho^{s/2+2\gamma+1}\right), \\
\dot{\psi}_\beta^* &= \varkappa\rho^{s/2+\gamma}\sum_{h=1}^{q-1} B_{\beta h} + \mathcal{O}\left(\rho^{s/2+2\gamma+1}\right), \\
\dot{\theta}_\nu^* &= \varkappa\rho^{s/2+\gamma}\sum_{i=1}^{\mu} B_{n+\nu,n+i} + \mathcal{O}\left(\rho^{s/2+2\gamma+1}\right).
\end{aligned}$$

Thus, taking into account the previous study, we verify that on the surface of the cone K_2 we have

$$\begin{aligned}
\dot{W}_\beta^0 &= 2\varkappa\rho^{s/2+2\gamma+1}\left[\sum_{h=1}^{q-1} B_{\beta h} - 2(1+\gamma)\right] + \mathcal{O}\left(\rho^{s/2+2\gamma+\frac{3}{2}}\right), \\
\dot{W}_{n+\nu}^0 &= 2\varkappa\rho^{s/2+2\gamma+1}\left[\sum_{i=1}^{\mu} B_{n+\nu,n+i} - 2(1+\gamma)\right] + \mathcal{O}\left(\rho^{s/2+2\gamma+\frac{3}{2}}\right), \\
\dot{W}_\alpha^0 &= -4\varkappa(1+\gamma)\rho^{s/2+2\gamma+1} + \mathcal{O}(\rho^{s/2+2\gamma+\frac{3}{2}}).
\end{aligned}$$

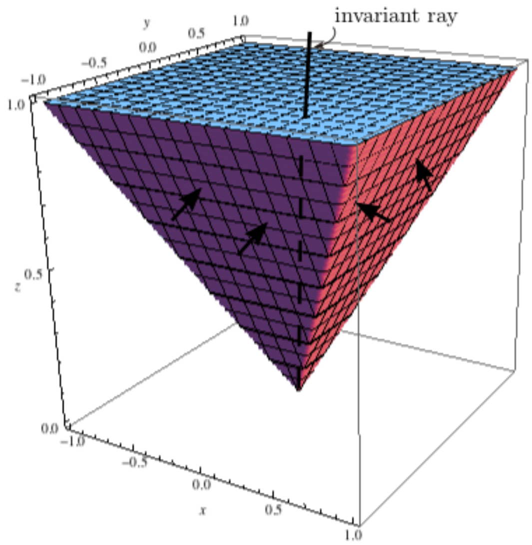


Figure 3.2: Representation of the cone $\Omega = \{V > 0\} \cap K_1$.

At this stage, we need to choose γ in a convenient way. More precisely, we choose $\gamma \in \mathbb{N} \cup \{0\}$ such that

$$\gamma = \left[\max_{\substack{\beta \in \{1,\dots,q-1\} \\ \nu \in \{1,\dots,\mu\}}} \left\{ \frac{1}{2} \sum_{h=1}^{q-1} B_{\beta h} - 1, \frac{1}{2} \sum_{i=1}^{\mu} B_{n+\nu,n+i} - 1 \right\} \right]. \tag{3.52}$$

With this election of γ, it follows that $\dot{W}_i^0 < 0$, for all $i = 1, \dots, n+\mu$ $(i \neq q)$. Thus, for the Chetaev function $V = \rho$, we have proved that the derivative of V along the solutions of the system (3.44) is positive definite inside the cone Ω and that trajectories can only enter through the walls or boundary of the cone. Hence, by applying Chetaev's Theorem 1.33, the equilibrium solution $(0, 0)$ of the complete Hamiltonian system (3.35) is unstable in the Liapunov sense. ■

Corollary 3.11. *Assume that the vectors of resonances do not have interactions* *and set* $\gamma \in \mathbb{N} \cup \{0\}$ *such that*

$$\gamma = \left[\max_{\substack{\beta \in \{1,\ldots,q-1\} \\ \nu \in \{1,\ldots,\mu\}}} \left\{ \frac{1}{2} \sum_{h=1}^{q-1} B_{\beta h} - 1, \frac{1}{2} B_{n+\nu,n+\nu} - 1 \right\} \right]. \tag{3.53}$$

Under the conditions of Corollary 3.5 and assuming that $\det(B_{\nu\zeta} - NI) \neq 0,\ N = 1, 2, \ldots, 2(1+\gamma)$ $(\nu, \zeta = 1, \ldots, n+\nu; \nu, \zeta \neq q, \ldots, n)$ *(I is the identity matrix), the origin of the complete Hamiltonian system* (3.22) *is unstable in the Liapunov sense.*

Proof. Since the μ vectors of resonance do not have interactions, then the matrix $B_{\nu\zeta}$ in (3.50) assumes the simplified form

$$B_{\nu\zeta} = \left(\begin{array}{ccc|ccc} B_{11} & \ldots & B_{1,q-1} & & & \\ \vdots & \ddots & \vdots & & \mathbf{0} & \\ B_{q-1,1} & \ldots & B_{q-1,q-1} & & & \\ \hline & & & B_{n+1,n+1} & \ldots & 0 \\ & \mathbf{0} & & \vdots & \ddots & \vdots \\ & & & 0 & \ldots & B_{n+\mu,n+\mu} \end{array} \right). \tag{3.54}$$

So, the Hamiltonian system (3.42) takes the form

$$\begin{aligned} \dot{\rho} &= 2\varkappa\rho^{s/2} + \mathcal{R}_0^*, & \dot{\psi}_\beta^* &= \varkappa\rho^{s/2-1}\sum_{h=1}^{q-1} B_{\beta h}\psi_h^* + \mathcal{R}_\beta^*, \\ \dot{I}_\alpha &= \mathcal{R}_\alpha^*, & \dot{\theta}_\nu^* &= \varkappa\rho^{s/2-1} B_{n+\nu,n+\nu}\theta_\nu^* + \mathcal{R}_{n+\nu}^*, \end{aligned}$$

with terms of higher order as in (3.45) and must be chosen $\gamma \in \mathbb{N} \cup \{0\}$ of the form (3.53). ■

3.2.3 Examples in systems with three, four and six degrees of freedom

Example with 3 degrees of freedom. Let us consider the case of two resonance vectors of order 3, say $k_1 = (1, -1, -1)$ and $k_2 = (2, 1, 0)$, with interaction between two frequencies, the Hamiltonian function in its Lie normal form is

$$H = \sum_{j=1}^{3} \omega_j I_j + 2A_1\sqrt{I_1 I_2 I_3}\cos(\varphi_1-\varphi_2-\varphi_3) + 2A_2\sqrt{I_1^2 I_2}\cos(2\varphi_1+\varphi_2) + \dots \quad (3.55)$$

So the associated model Hamiltonian system is

$$\begin{aligned}
\dot{I}_1 &= -2A_1\sqrt{I_1 I_2 I_3}\sin(\varphi_1-\varphi_2-\varphi_3) - 4A_2\sqrt{I_1^2 I_2}\sin(2\varphi_1+\varphi_2),\\
\dot{I}_2 &= 2A_1\sqrt{I_1 I_2 I_3}\sin(\varphi_1-\varphi_2-\varphi_3) - 2A_2\sqrt{I_1^2 I_2}\sin(2\varphi_1+\varphi_2),\\
\dot{I}_3 &= 2A_1\sqrt{I_1 I_2 I_3}\sin(\varphi_1-\varphi_2-\varphi_3),\\
\dot{\varphi}_1 &= -\omega_1 - \frac{A_1 I_2 I_3}{\sqrt{I_1 I_2 I_3}}\cos(\varphi_1-\varphi_2-\varphi_3) - \frac{2A_2 I_2}{\sqrt{I_2}}\cos(2\varphi_1+\varphi_2),\\
\dot{\varphi}_2 &= -\omega_1 - \frac{A_1 I_2 I_3}{\sqrt{I_1 I_2 I_3}}\cos(\varphi_1-\varphi_2-\varphi_3) - \frac{A_2 I_1}{\sqrt{I_2}}\cos(2\varphi_1+\varphi_2),\\
\dot{\varphi}_3 &= 3\omega_1 - \frac{A_1 I_2}{\sqrt{I_1 I_2 I_3}}\cos(\varphi_1-\varphi_2-\varphi_3).
\end{aligned}$$

If $A_1 \neq 0$ and $A_2 \neq 0$, the invariant ray solution has the form

$$\begin{aligned}
I_1(t) &= \tfrac{1}{A_2\left(A_2-\sqrt{A_2^2-A_1^2}\right)+A_1^2} b(t), \quad I_2(t) = -\tfrac{1}{2A_2\left(\sqrt{A_2^2-A_1^2}-A_2\right)+A_1^2} b(t),\\
I_3(t) &= -\tfrac{A_1^2}{A_1^4+A_1^2A_2\left(\sqrt{A_2^2-A_1^2}+A_2\right)+4A_2^3\left(\sqrt{A_2^2-A_1^2}-A_2\right)} b(t),\\
\theta_1^0 &= -\tfrac{\pi}{2}\tfrac{A_1}{|A_1|}, \quad \theta_2^0 = \tfrac{\pi}{2}\tfrac{A_2}{|A_2|}, \quad b(t) = \tfrac{4}{(t+c)^2},
\end{aligned}$$

under the condition $0 < |A_1/A_2| < 1$. In this case the matrix $B_{\nu\zeta}$ in (3.50) is given by

$$B_{\nu\zeta} = \begin{pmatrix} B_{11} & B_{12} & 0 & 0\\ B_{21} & B_{22} & 0 & 0\\ 0 & 0 & B_{44} & B_{45}\\ 0 & 0 & B_{54} & B_{55} \end{pmatrix},$$

where $\delta = \sqrt{A_2^2 - A_1^2}$ and

$$
\begin{aligned}
B_{11} &= -\frac{A_2\big(7A_1^2+4A_2(A_2-\delta)\big)}{2\big(A_1^2+A_2(A_2-\delta)\big)\sqrt{2A_2(A_2-\delta)-A_1^2}} - 2,\\
B_{12} &= -\frac{A_2\big(7A_1^2+4A_2(A_2-\delta)\big)}{3\sqrt{2}\sqrt{-\big(A_1^2+A_2(A_2-\delta)\big)^2\big(A_1^2+2A_2(\delta-A_2)\big)}},\\
B_{21} &= \frac{3A_2\big(A_1^2+2A_2(\delta-A_2)\big)}{\sqrt{2}\sqrt{-\big(A_1^2+A_2(A_2-\delta)\big)^2\big(A_1^2+2A_2(\delta-A_2)\big)}},\\
B_{22} &= -\frac{\sqrt{2A_2(A_2-\delta)-A_1^2}\,A_2}{A_1^2+A_2(A_2-\delta)} - 2,\\
B_{44} &= -\frac{3A_1^4+2A_2\delta A_1^2+4A_2^3(\delta-A_2)}{\sqrt{A_1^8+A_2(A_2+2\delta)A_1^6+2A_2^3(5\delta-7A_2)A_1^4-16A_2^6A_1^2+32A_2^7(A_2-\delta)}},\\
B_{45} &= \frac{A_2\big(A_1^2+4A_2(A_2-\delta)\big)}{\big(A_1^2+A_2(A_2-\delta)\big)\sqrt{2A_2(A_2-\delta)-A_1^2}},\\
B_{54} &= -\frac{3A_1^4}{\sqrt{A_1^8+A_2(A_2+2\delta)A_1^6+2A_2^3(5\delta-7A_2)A_1^4-16A_2^6A_1^2+32A_2^7(A_2-\delta)}},\\
B_{55} &= \frac{3A_2\big(A_1^2+2A_2(A_2-\delta)\big)}{\big(A_1^2+A_2(A_2-\delta)\big)\sqrt{2A_2(A_2-\delta)-A_1^2}}.
\end{aligned}
$$

Then just choose $\gamma \in \mathbb{N} \cup \{0\}$, such that

$$
\gamma = \left[\max_{\substack{\beta \in \{1,2\} \\ \nu \in \{1,2\}}} \left\{ \frac{1}{2}\sum_{h=1}^{2} B_{\beta h} - 1, \frac{1}{2}\sum_{i=1}^{2} B_{3+\nu,3+i} - 1 \right\} \right].
$$

We verified that $\det(B_{\nu\zeta} - NI) \neq 0$, for $N = 1, \ldots, 2(1+\gamma)$ when $0 < |A_1/A_2| < 1$. Hence, by Theorem 3.4 the null solution of the complete Hamiltonian system associated with the Hamiltonian function (3.55) is unstable in the Liapunov sense whenever $0 < |A_1/A_2| < 1$.

Example with 4 degrees of freedom. Let us consider the case of two resonance vectors of order 3, say $k_1 = (2,1,0,0)$ and $k_2 = (0,0,2,1)$, without interaction between their frequencies. Whose Hamiltonian function in its Lie normal form is

$$
H = \sum_{j=1}^{4}\omega_j I_j + 2A_1\sqrt{I_1^2 I_2}\cos(2\varphi_1+\varphi_2) + 2A_2\sqrt{I_3^2 I_4}\cos(2\varphi_3+\varphi_4) + \ldots. \tag{3.56}
$$

So the associated model Hamiltonian system is

$$\begin{aligned}
\dot{I}_1 &= -4A_1\sqrt{I_1^2 I_2}\sin(2\varphi_1+\varphi_2), & \dot{\varphi}_1 &= -\omega_1 - \frac{2A_1 I_2}{\sqrt{I_2}}\cos(2\varphi_1+\varphi_2),\\
\dot{I}_2 &= -2A_1\sqrt{I_1^2 I_2}\sin(2\varphi_1+\varphi_2), & \dot{\varphi}_2 &= -\omega_2 - \frac{A_1 I_1}{\sqrt{I_2}}\cos(2\varphi_1+\varphi_2),\\
\dot{I}_3 &= -4A_2\sqrt{I_3^2 I_4}\sin(2\varphi_3+\varphi_4), & \dot{\varphi}_3 &= -\omega_3 - \frac{2A_2 I_4}{\sqrt{I_4}}\cos(2\varphi_3+\varphi_4),\\
\dot{I}_4 &= -2A_2\sqrt{I_3^2 I_4}\sin(2\varphi_3+\varphi_4), & \dot{\varphi}_4 &= -\omega_4 - \frac{A_2 I_3}{\sqrt{I_4}}\cos(2\varphi_3+\varphi_4).
\end{aligned}$$

If $A_1 \neq 0$ and $A_2 \neq 0$, the invariant ray solution has the form

$$\begin{aligned}
I_1(t) &= \tfrac{1}{2A_1^2}b(t), \quad I_2(t) = \tfrac{1}{4A_1^2}b(t), \quad I_3(t) = \tfrac{1}{2A_2^2}b(t), \quad I_4(t) = \tfrac{1}{4A_2^2}b(t),\\
\theta_1^0 &= -\tfrac{\pi}{2}\tfrac{A_1}{|A_1|}, \quad \theta_2^0 = \tfrac{\pi}{2}\tfrac{A_2}{|A_2|}, \quad b(t) = \tfrac{4}{(t+c)^2}.
\end{aligned}$$

In this case the matrix $B_{\nu\zeta}$ in (3.54) is given by

$$B_{\nu\zeta} = \begin{pmatrix}
-\frac{5}{3} & -\frac{1}{2\sqrt{6}} & \frac{1}{2\sqrt{3}} & 0 & 0\\
\frac{2\sqrt{\frac{2}{3}}}{3} & -\frac{7}{3} & \frac{\sqrt{2}}{3} & 0 & 0\\
0 & 0 & -2 & 0 & 0\\
0 & 0 & 0 & -3 & 0\\
0 & 0 & 0 & 0 & 3
\end{pmatrix}.$$

Thus choosing $\gamma = 0$ and observing that $\det(B_{\nu\zeta} - NI) \neq 0$, for $N = 1, 2, \ldots$, then by Corollary 3.11 the null solution of the complete Hamiltonian system associated with the Hamiltonian function (3.56) is unstable in the Liapunov sense.

Example with 6 degrees of freedom. Let us consider the case of three resonance vectors of order $n+1$, such that n is even, say $P_1 = (n, 1, 0, 0, 0, 0)$, $P_2 = (0, 0, n, 1, 0, 0)$ and $P_3 = (0, 0, 0, 0, n, 1)$, without interaction between their frequencies, whose Hamiltonian function in its Lie normal form is

$$\begin{aligned}
H &= \sum_{j=1}^{6}\omega_j I_j + 2A_1\sqrt{I_1^n I_2}\cos(n\varphi_1+\varphi_2) + 2A_2\sqrt{I_3^n I_4}\cos(n\varphi_3+\varphi_4) + \\
&\quad 2A_3\sqrt{I_5^n I_6}\cos(n\varphi_5+\varphi_6) + \ldots .
\end{aligned} \tag{3.57}$$

So the associated model Hamiltonian system is

$$
\begin{aligned}
\dot{I}_1 &= -2nA_1\sqrt{I_1^n I_2}\sin(n\varphi_1+\varphi_2), & \dot{\varphi}_1 &= -\omega_1 - \frac{nA_1 I_1^{n-1} I_2}{\sqrt{I_1^n I_2}}\cos(n\varphi_1+\varphi_2),\\
\dot{I}_2 &= -2A_1\sqrt{I_1^n I_2}\sin(n\varphi_1+\varphi_2), & \dot{\varphi}_2 &= -\omega_2 - \frac{A_1 I_1^{n}}{\sqrt{I_1^n I_2}}\cos(n\varphi_1+\varphi_2),\\
\dot{I}_3 &= -2nA_2\sqrt{I_3^n I_4}\sin(n\varphi_3+\varphi_4), & \dot{\varphi}_3 &= -\omega_3 - \frac{nA_2 I_3^{n-1} I_4}{\sqrt{I_3^n I_4}}\cos(n\varphi_3+\varphi_4),\\
\dot{I}_4 &= -2A_2\sqrt{I_3^n I_4}\sin(n\varphi_3+\varphi_4), & \dot{\varphi}_4 &= -\omega_4 - \frac{A_2 I_3^{n}}{\sqrt{I_3^n I_4}}\cos(n\varphi_3+\varphi_4),\\
\dot{I}_5 &= -2nA_3\sqrt{I_5^n I_6}\sin(n\varphi_5+\varphi_6), & \dot{\varphi}_5 &= -\omega_5 - \frac{nA_3 I_5^{n-1} I_6}{\sqrt{I_5^n I_6}}\cos(n\varphi_5+\varphi_6),\\
\dot{I}_6 &= -2A_3\sqrt{I_5^n I_6}\sin(n\varphi_5+\varphi_6), & \dot{\varphi}_6 &= -\omega_6 - \frac{A_3 I_5^{n}}{\sqrt{I_5^n I_6}}\cos(n\varphi_5+\varphi_6).
\end{aligned}
$$

If $A_1 \neq 0$, $A_2 \neq 0$ and $A_3 \neq 0$, the invariant ray solution has the form

$$
\begin{aligned}
I_1(t) &= \tfrac{1}{n^{1/(n-1)}A_1^{2/(n-1)}}b(t),\ I_2(t) = \tfrac{1}{n^{n/(n-1)}A_1^{2/(n-1)}}b(t), I_3(t) = \tfrac{1}{n^{1/(n-1)}A_2^{2/(n-1)}}b(t),\\
I_4(t) &= \tfrac{1}{n^{n/(n-1)}A_2^{2/(n-1)}}b(t),\ I_5(t) = \tfrac{1}{n^{1/(n-1)}A_3^{2/(n-1)}}b(t),\ I_6(t) = \tfrac{1}{n^{n/(n-1)}A_3^{2/(n-1)}}b(t),\\
\theta_1^0 &= -\tfrac{\pi}{2}\tfrac{A_1}{|A_1|},\ \theta_2^0 = -\tfrac{\pi}{2}\tfrac{A_2}{|A_2|},\ \theta_3^0 = -\tfrac{\pi}{2}\tfrac{A_3}{|A_3|},\\
b(t) &= 2^{2/(n-1)}\left[-(n-1)\left(c_1+2t\right)\right]^{-2/(n-1)}.
\end{aligned}
$$

In this case the matrix $B_{\nu\zeta}$ in (3.54) has the form

$$
B_{\nu\zeta} = \begin{pmatrix}
B_{11} & B_{12} & B_{13} & B_{14} & B_{15} & 0 & 0 & 0\\
B_{21} & B_{22} & B_{23} & B_{24} & B_{25} & 0 & 0 & 0\\
B_{31} & B_{32} & B_{33} & B_{34} & B_{35} & 0 & 0 & 0\\
B_{41} & B_{42} & B_{43} & B_{44} & B_{45} & 0 & 0 & 0\\
B_{51} & B_{52} & B_{53} & B_{54} & B_{55} & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & B_{77} & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & B_{88} & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & B_{99}
\end{pmatrix},
$$

with $B_{11} = \frac{2}{15}(5n-16)$, $B_{12} = \frac{4}{3\sqrt{5}}$, $B_{13} = -\frac{4n}{3\sqrt{15}}, B_{14} = -\frac{n+3}{3\sqrt{10}}$, $B_{15} = \frac{n-1}{3\sqrt{5}}$, $B_{21} = \frac{2(5n-1)}{5\sqrt{5}}$, $B_{22} = -\frac{6}{5}$, $B_{23} = -\frac{4n}{5\sqrt{3}}$, $B_{24} = -\frac{n+3}{5\sqrt{2}}$, $B_{25} = \frac{n-1}{5}$, $B_{31} =$

$-\frac{n+1}{\sqrt{15}}$, $B_{32} = -\frac{n+1}{2\sqrt{3}}$, $B_{33} = \frac{n-7}{3}$, $B_{34} = -\frac{n-1}{2\sqrt{6}}, B_{35} = 0, B_{41} = -\frac{2}{3}\sqrt{\frac{2}{5}}(n+1), B_{42} = -\frac{1}{3}\sqrt{2}(n+1)$, $B_{43} = \frac{2}{3}\sqrt{\frac{2}{3}}(n-1)$, $B_{44} = -\frac{1}{3}(n+5)$, $B_{45} = \frac{1}{3}\sqrt{2}(n-1)$, $B_{51} = 0$, $B_{52} = 0$, $B_{53} = 0$, $B_{54} = 0$, $B_{55} = -2$, $B_{77} = -n-1$, $B_{88} = -n-1$, $B_{99} = n+1$. Thus, by Corollary 3.11 we must choose γ such that $\gamma = \max_{\beta \in \{1,\ldots,5\}} \left\{ \frac{1}{2} \sum_{h=1}^{5} B_{\beta h} - 1 \right\}$, and it is characterized in the Table 3.1 as function of n.

$l = \frac{n}{2}$	2	3	4	5	6	7	8	9	10	...
order of resonance	5	7	9	11	13	15	17	19	21	...
γ	0	0	0	1	1	2	2	3	3	...

Table 3.1: Values of γ, according to the order of resonance.

In the previous cases we verified that $\det(B_{\nu\zeta} - NI) \neq 0$, for $N = 1, \ldots 2(1+\gamma)$. Since the determinant is zero in $N = 1+n$ and $N = -1+n$. Then, by Corollary 3.11 the null solution of the complete Hamiltonian system associated with the Hamiltonian function (3.57) is unstable in the sense of Liapunov. Note that the Corollary 3.11 does not apply to $l = 1$, that is, three resonance vectors of order 3, since the determinant is null at $N = 1$.

Chapter 4

Application to the spatial restricted circular three-body problem

In this chapter we apply our main stability and instability results to study the Lagrangian point L_4 (or L_5) in the spatial restricted circular three-body problem. First, we characterize the resonance type according to the parameters and classify the resonances according to the dimension of the set S. Next, we normalize the quadratic part around L_4 and determine the Lie normal form of the Hamiltonian to study the nonlinear stability. Specifically, we study stability using the result of Chapter 2 and instability using the results of Chapter 3.

4.1 Statement of the problem

The Hamiltonian function associated to this problem in rotating coordinates (x, y, z, X, Y, Z) is (see for example [51] or [71] for details)

$$\begin{aligned}\mathcal{H} &= \tfrac{1}{2}\left(X^2+Y^2+Z^2\right)+Xy-xY-\tfrac{1}{2}(1-\mu)\mu- \\ &\quad \tfrac{\mu}{\sqrt{(\mu+x-1)^2+y^2+z^2}}-\tfrac{1-\mu}{\sqrt{(\mu+x)^2+y^2+z^2}}.\end{aligned} \tag{4.1}$$

The Hamiltonian system associated to (4.1) is

$$\begin{aligned} \dot{x} &= \mathcal{H}_X, & \dot{X} &= -\mathcal{H}_x, \\ \dot{y} &= \mathcal{H}_Y, & \dot{Y} &= -\mathcal{H}_y, \\ \dot{z} &= \mathcal{H}_Z, & \dot{Z} &= -\mathcal{H}_z, \end{aligned} \tag{4.2}$$

which represents an autonomous system with three degrees of freedom depending on the parameter $\mu \in (0, 1/2]$. The coordinates of L_4 and L_5 in the six-dimensional phase space are

$$\left(1/2-\mu, \pm\sqrt{3}/2, 0, \mp\sqrt{3}/2, 1/2-\mu, 0\right),$$

where the upper signs apply for L_4 and the lower signs for L_5.

We propose to study the type of stability of the equilibrium point L_4 and L_5. It is known that the type of stability of L_5 is the same as L_4, so from now on that we are going to study the point L_4.

We start moving the equilibrium solution L_4 at the origin by means of the following change of coordinates $x = \varepsilon x_1 + 1/2 - \mu$, $y = \varepsilon y_1 + \sqrt{3}/2$, $z = \varepsilon z_1$, $X = \varepsilon X_1 - \sqrt{3}/2$, $Y = \varepsilon Y_1 + 1/2 - \mu$, $Z = \varepsilon Z_1$ and we expand the Hamiltonian function (4.1) in Taylor series in a neighborhood of the origin, so the Hamiltonians H_j with $j = 2, 3, 4$ are

$$\begin{aligned}
H_2 &= \tfrac{1}{8}\left(2x_1\left(3\sqrt{3}(2\mu-1)y_1 - 4Y_1\right) + x_1^2 + 8X_1y_1 + 4X_1^2 - 5y_1^2 + 4\left(Y_1^2 + z_1^2 + Z_1^2\right)\right),\\
H_3 &= \tfrac{1}{16}\left(7(2\mu-1)x_1^3 - 3(2\mu-1)x_1\left(11y_1^2 - 4z_1^2\right) + 3\sqrt{3}x_1^2y_1 + 3\sqrt{3}y_1\left(y_1^2 - 4z_1^2\right)\right),\\
H_4 &= \tfrac{1}{128}\left(100\sqrt{3}(1-2\mu)x_1^3y_1 + 60\sqrt{3}(2\mu-1)x_1y_1\left(3y_1^2 - 4z_1^2\right) + 6x_1^2\left(4z_1^2 - 41y_1^2\right) + \right.\\
&\left. 37x_1^4 - 3\left(-88y_1^2z_1^2 + y_1^4 + 16z_1^4\right)\right).
\end{aligned}$$

Next, we observe that the linearization matrix associated to this equilibrium is

$$B = \begin{pmatrix}
0 & 1 & 0 & 1 & 0 & 0\\
-1 & 0 & 0 & 0 & 1 & 0\\
0 & 0 & 0 & 0 & 0 & 1\\
-\frac{1}{4} & \frac{3}{4}\sqrt{3}(1-2\mu) & 0 & 0 & 1 & 0\\
\frac{3}{4}\sqrt{3}(1-2\mu) & \frac{5}{4} & 0 & -1 & 0 & 0\\
0 & 0 & -1 & 0 & 0 & 0
\end{pmatrix}.$$

The pure imaginary eigenvalues of matrix B are $\pm\lambda_1, \pm\lambda_2$ and $\pm\lambda_3$ whenever $0 < \mu \leq \mu_R = \frac{1}{2}(1 - \sqrt{69}/9)$ (called the Routh's value) with

$$i\omega_1 = i\frac{\sqrt{1+\sqrt{1-27\mu-27\mu^2}}}{\sqrt{2}}, \quad i\omega_2 = i\frac{\sqrt{1-\sqrt{1-27\mu-27\mu^2}}}{\sqrt{2}}, \quad \lambda_3 = i.$$

Note that $0 < \omega_2 < \sqrt{2}/2 < \omega_1 < 1$, $\omega_1^2 + \omega_2^2 = 1$, $\omega_3 = 1$. It is verified that when $\mu > \mu_R$ the equilibria are of focus-center type, therefore unstable as they come from a symplectic system, so we restrict μ to $(0, \mu_R]$. Moreover in $(0, \mu_R)$ the corresponding eigenvectors form a basis of $\mathbb{R}^6$, so using Markeev's procedure for the normalization of the quadratic part (see [51]), we determine that the quadratic part assumes the form

$$H_2 = \omega_1 I_1 - \omega_2 I_2 + \omega_3 I_3.$$

In particular, in the interval $(0, \mu_R)$ the quadratic part is of indefinite sign.

We observe that $0 < \omega_1/\omega_2 < 1$ and $1 < \omega_3/\omega_2 < \sqrt{2}$. There are an infinite number of values of μ such that $\omega_1/\omega_2 = r \in \mathbb{Q}$ with $r \in (0,1)$ and $\omega_3/\omega_2 = 1/s \in \mathbb{Q}$ with $s \in (1/\sqrt{2}, 1)$ given by

$$\mu_r = \frac{1}{2} - \frac{\sqrt{27r^4 + 38r^2 + 27}}{6\sqrt{3}(r^2+1)}, \quad \mu_s = \frac{1}{2} - \frac{\sqrt{48s^4 - 48s^2 + 81}}{18}. \tag{4.3}$$

It is clear that the planar case $x_3 = y_3 = 0$ is an invariant problem of the spatial restricted circular three body problem. In this particular case we know that the stability of the equilibrium L_4 is completed solved. In fact, in Table 4.1 we summarize the results about the type of stability of the triangular points according the parameter μ, note that $\mu_1 < \mu_2 < \mu_3 < \mu_R$.

Value μ	Type of stability	Author, (Year)
$\mu \in (0, \mu_1)$, $\mu \neq \mu_2, \mu_3, \mu_R$	stable	Deprit, A., Deprit-Bartolomé (1967)
$\mu = \mu_{(1,2,0)}$	unstable	Markeev, (1966)
$\mu = \mu_{(1,3,0)}$	unstable	Markeev, (1966)
$\mu = \mu_1 \approx 0.0109137$	stable	Markeev (1969)
$\mu = \mu_R$	stable	Sokolskii, (1978)
$\mu \in (\mu_R, 1/2]$	unstable	exist eigenvalue λ, with $\mathrm{Re}(\lambda) \neq 0$.

Table 4.1: Type of stability of the Lagrangian points depending on μ.

Observing Table 4.1 we deduce that the equilibrium point L_4 (or L_5) is Lyapunov stable for every $\mu \in (0, \mu_R]$ excepting at two values $\mu = \mu_{(1,2,0)}$ and $\mu = \mu_{(1,3,0)}$. These values correspond to resonance of order three and four, respectively.

For the spatial case the study of stability of the Lagrangian equilibria, in general, still is an open problem. As it is said by Celletti and Giorgilli in [17] "the same methods however do not allow to draw definite conclusions in the spatial case, since the existence of many invariant tori, which could still be proven, does not prevent an orbit starting in a gap between tori from diffusing through the gaps, and going far from the equilibrium: this is the so called Arnold diffusion. Thus, the stability of the equilibrium can only be guaranteed from the viewpoint of measure theory, since the majority of initial data lies on invariant tori". In the following Table 4.2 we summarize some results for this point.

Value μ	Type of stability	Author, (Year)
$\mu \in (\mu_R, 1/2]$	Unstable	Exist eigenvalue λ, with $Re(\lambda) \neq 0$
$\mu = \mu_{(1,2,0)}$	Unstable	Markeev, (1966)
$\mu = \mu_{(1,3,0)}$	Unstable	Markeev, (1966)
$\mu \neq \mu_{(1,2,0)}, \mu_{(1,3,0)}$	S.M.I.C.	Markeev, (1972)
$\mu \in (\mu_{(1,2,0)}, \mu_R)$	Formally stable	Markeev, (1973)
$\mu \in I$	Formally stable	Markeev, (1978)
$\mu = \mu_1 \approx 0.0109137$	Stable	Markeev (1969)
$\mu = \mu_R$	Formally stable	Markeev, (1978)
$\mu \in (0, \mu_R)$, excepting for a few values of μ that lead to resonances	Nekhoroshev stability	Delshams, Giorgilli, Fontich, Galgani, Simò, (1989)
$\mu \in ?$	Nekhoroshev stability?	Celletti, Giorgilli, (1991)
$\mu \in (\mu_1, \mu_2)$, $\mu \neq \mu_{(1,3,0)}, \mu_3$, $\mu \neq \mu_{(3,3,-2)}, \mu_{(0,3,1)}$	Nekhoroshev stability (3-jet)	Benettin, Fassò, Guzzo, (1998)
$\mu \in (0, \mu_1) \cup (\mu_2, \mu_R)$, $\mu \neq \mu_{(1,2,0)}$	Nekhoroshev stability (D.Q.C.)	Benettin, Fassò, Guzzo, (1998)
$\mu \in (0, \mu_R)$ $\mu \neq \mu_r, \mu_s$	Normally stable	Meyer, Palacián, Yanguas, (2013)
$\mu = \mu_3$	Nekhoroshev stability (steep)	Schirinzi, Guzzo, (2015)

Table 4.2: Type of stability of the Lagrangian points depending on μ. S.M.I.C. means stable for most initial conditions (in the sense of Lebesgue measure). $I = (0.010913, 0.016376) \setminus \mu_{(1,3,0)}$ eliminating the case of double resonance of order ≥ 7. $\mu_1 = 0.0109137$. $\mu_2 = 0.0163768$. $\mu_3 = 0.0147808$. D.Q.C. means directionally quasi-convex.

4.2 Analysis of the dimension of S

We remark that ω_1 and ω_2 are rationally independent excluding the values $\mu = \mu_r$ while ω_2 and ω_3 are rationally independent excluding the values $\mu = \mu_s$. The three frequencies can be rationally dependent when $\mu_r = \mu_s$ and s and r rational numbers. It occurs for $s = 1/\sqrt{r^2+1}$, $r = m/n$ with m and n integers such that $\sqrt{m^2+n^2}$ is also an integer. For instance, if $m = 3$ and $n = 4$ then $r = 3/4$, $s = 4/5$ and $(\omega_1, \omega_2, \omega_3) = (4/5, 3/5, 1)$ but if $m = 5$ and $n = 12$ then $r = 5/12$, $s = 12/13$ and $(\omega_1, \omega_2, \omega_3) = (5/13, 12/13, 1)$. In general the dimension of S in this problem can be 0, 1 or 2. Next, we proceed to study the different situations.

- (a) Suppose $\omega_1 \in \mathbb{Q}$.
  - (a_1) If $\omega_2 \in \mathbb{R} \setminus \mathbb{Q}$, then $F_1 = \omega_1 I_1 + I_3$ and $F_2 = I_2$. Thus, $S = \{(I_1, I_2, I_3) : \omega_1 I_1 + I_3 = 0, I_2 = 0\} = \{0\}$.
  - (a_2) If $\omega_2 \in \mathbb{Q}$, then $F = \omega_1 I_1 - \omega_2 I_2 + I_3$. Thus, $S = \{(I_1, I_2, I_3) : \omega_1 I_1 - \omega_2 I_2 + I_3 = 0\} = \{(I_1, I_2, -\omega_1 I_1 + \omega_2 I_2), I_1, I_2 \geq 0\}$. Therefore, $\dim S = 2$.
- (b) Suppose $\omega_1 \in \mathbb{R} \setminus \mathbb{Q}$.
  - (b_1) If $\omega_2 \in \mathbb{R}\setminus\mathbb{Q}$ and $\omega_1/\omega_2 \in \mathbb{R}\setminus\mathbb{Q}$, then there are not resonant relations. Thus, $S = \{0\}$.
  - (b_2) If $\omega_2 \in \mathbb{R} \setminus \mathbb{Q}$ and $\omega_1/\omega_2 \in \mathbb{Q}$, then $F_1 = -\omega_1 I_1 + \omega_2 I_2$, $F_2 = I_3$. Thus, $S = \{(I_1, I_2, I_3) : -\omega_1 I_1 + \omega_2 I_2 = 0, I_3 = 0\} = \{(\omega_2 I_1, \omega_1 I_1, 0), I_1 \geq 0\}$. Therefore, $\dim S = 1$.
  - (b_3) If $\omega_2 \in \mathbb{Q}$, then $F_1 = -\omega_2 I_2 + I_3$, $F_2 = I_1$. Thus, $S = \{(I_1, I_2, I_3) : -\omega_2 I_2 + I_3 = 0, I_1 = 0\} = \{(0, I_2, \omega_2 I_2), I_2 \geq 0\}$. Therefore, $\dim S = 1$.

Now we see that in the case

- (a_1) the vector of resonance is $k_1 = (1, 0, -\omega_1)$, since the components of vector of resonance change of sign applying Theorem 1.27 we obtain that the equilibrium is Lie stable.
- (a_2) there are multiple resonance given by the vector $k_1 = (1, 0, -\omega_1)$ and $k_2 = (0, 1, \omega_2)$.
- (b_1) since there are not resonance by Remark 2.4.2 follows that the equilibrium is Lie stable.
- (b_2) the vector of resonance is given by $k = (\omega_2, \omega_1, 0)$, here we can have stability or instability.
- (b_3) the vector of resonance is given by $k = (0, 1, \omega_2)$, here we can have stability or instability.

Therefore, in order to describe the type of stability of the point equilibrium L_4 the cases are still missing the cases $\dim S = 1$ and $\dim S = 2$. Assuming are not resonance up to order four, the Lie normal form up order four is

$$\mathcal{H} = H_2 + \mathcal{H}_4 + \ldots, \tag{4.4}$$

where $\mathcal{H}_4 = c_{200}I_1^2 + c_{110}I_1I_2 + c_{101}I_1I_3 + c_{020}I_2^2 + c_{011}I_2I_3 + c_{002}I_3^2$ and

$$\begin{aligned}
c_{200} &= \tfrac{\omega_2^2(124\omega_1^4-696\omega_1^2+81)}{144(1-2\omega_1^2)^2(1-5\omega_1^2)}, \\
c_{110} &= -\tfrac{\omega_1\omega_2(64\omega_1^2\omega_2^2+43)}{6(1-2\omega_1^2)(1-2\omega_2^2)(1-5\omega_1^2)(1-5\omega_2^2)}, \\
c_{101} &= -\tfrac{8\omega_1\omega_2^2}{3(1-2\omega_1^2)(4-\omega_1^2)}, \\
c_{020} &= \tfrac{\omega_1^2(124\omega_2^4-696\omega_2^3+81)}{144(1-2\omega_2^2)^2(1-5\omega_2^2)}, \\
c_{011} &= \tfrac{8\omega_2\omega_1^2}{3(1-2\omega_2^2)(4-\omega_2^2)}, \\
c_{002} &= -\tfrac{\omega_1^2\omega_2^2}{3(4-\omega_1^2)(4-\omega_2^2)}.
\end{aligned} \tag{4.5}$$

Here we do not include the expressions for the first term W_1 of the generating transformation.

Next, we analyze the veracity of the normalization up to order four. After a very carefully analysis of the denominator of the coefficients in $\mathcal{H}_4$, W_1 and taking into account that $0 < \omega_2 < \sqrt{2}/2 < \omega_1 < 1$ and $\omega_1^2 + \omega_2^2 = 1$. We arrive that the unique cases where the normal form is not valid until order four is when $\omega_1/\omega_2 = 2$ and $\omega_1/\omega_3 = 3$. In particular, the quartic term of the normalized Hamiltonian is as (4.4), which do not have resonant terms.

1. $\dim S = 1$

   (b_2) If $\omega_1, \omega_2 \in \mathbb{R} \setminus \mathbb{Q}, \omega_1/\omega_2 \in \mathbb{Q}$, then the equilibrium L_4 is Lie stable in the following intervals of ω_1:

   $$\begin{aligned}
   &\tfrac{\sqrt{2}}{2} < \omega_1 < \tfrac{2\sqrt{5}}{5}, \ \tfrac{2\sqrt{5}}{5} < \omega_1 < 0.935871439168618\ldots, \\
   &0.935871439168618\ldots < \omega_1 < 1,
   \end{aligned}$$

   except in the case of $\omega_1 = 3\sqrt{10}/10$.

   (b_3) If $\omega_1 \in \mathbb{R} \setminus \mathbb{Q}, \omega_2 \in \mathbb{Q}$, then the equilibrium L_4 is Lie stable in the following intervals of ω_1:

   $$\begin{aligned}
   &\tfrac{\sqrt{2}}{2} < \omega_1 < \tfrac{2\sqrt{5}}{5}, \ \tfrac{2\sqrt{5}}{5} < \omega_1 < \tfrac{1}{2}\sqrt{2 + \sqrt{\tfrac{2}{161}(-219 + \sqrt{199945})}}, \\
   &\tfrac{1}{2}\sqrt{2 + \sqrt{\tfrac{2}{161}(-219 + \sqrt{199945})}} < \omega_1 < 1.
   \end{aligned}$$

2. $\dim S = 2$

   (a_2) If $\omega_1, \omega_2 \in \mathbb{Q}$, then the equilibrium L_4 is Lie stable in the following intervals of ω_1:

$$\tfrac{\sqrt{2}}{2} < \omega_1 < \tfrac{2\sqrt{5}}{5},\ \omega_1 = \tfrac{n}{\sqrt{m^2+n^2}},\ \omega_2 = -\tfrac{m}{\sqrt{m^2+n^2}}$$

   with $n, m \in \mathbb{Z}^+$ such that $0 < m < n$ and $m^2 + n^2$ is a perfect square.

For special cases:

- $\omega_1 = \sqrt{2}/2,\ 3\sqrt{10}/10$, we obtain that the equilibrium is unstable in the Liapunov sense.
- $\omega_1 = \frac{1}{2}\sqrt{2 + \sqrt{\frac{2}{161}(-219 + \sqrt{199945})}},\ 0.935871439168618\ldots$, we obtain that the equilibrium is Lie stable.

4.3 Characterization of the resonances

We are going to analyze the curves of resonances of order $3, 4, \ldots, 8$ in the interval $0 < \mu < \mu_R$, i.e., $\omega_1 k_1 - \omega_2 k_2 + \omega_3 k_3 = 0, \quad |k| = |k_1| + |k_2| + |k_3| = j$. For each $j = 3, \ldots, 8$, we compute that in the interval $0 < \mu < \mu_R$ there are exactly 48 distinct single resonance of order j, that are shown in Table 4.3.

$\lvert k\rvert$	Vector	μ	$\lvert k\rvert$	Vector	μ
3	$(1,2,0)$	$\mu_{(1,2,0)} := \frac{45-\sqrt{1833}}{90}$	7	$(2,5,0)$	$\frac{261-\sqrt{63321}}{522}$
	$(0,2,1)$	$\frac{3-2\sqrt{2}}{6}$		$(2,4,1)$	$\frac{75-2\sqrt{6(214-3\sqrt{19})}}{150}$
4	$(2,1,-1)$	$\frac{75-\sqrt{4857}}{150}$		$(2,4,-1)$	$\frac{75-2\sqrt{6(214+3\sqrt{19})}}{150}$
	$(1,3,0)$	$\mu_{(1,3,0)} := \frac{15-\sqrt{213}}{30}$		$(1,6,0)$	$\frac{111-\sqrt{12129}}{222}$
	$(0,3,1)$	$\frac{81-\sqrt{6177}}{162}$		$(1,5,1)$	$\frac{507-\sqrt{237849}}{1014}$
5	$(2,3,0)$	$\frac{39-\sqrt{1329}}{78}$		$(1,4,2)$	$\frac{867-\sqrt{3(227459-3840\sqrt{13})}}{1734}$
	$(2,2,-1)$	$\frac{6-\sqrt{33}}{12}$		$(1,-4,-2)$	$\frac{867-\sqrt{3(227459+3840\sqrt{13})}}{1734}$
	$(1,4,0)$	$\frac{153-\sqrt{22641}}{306}$		$(0,6,1)$	$\frac{81-2\sqrt{1614}}{162}$
	$(1,3,1)$	$\frac{75-\sqrt{4857}}{150}$		$(0,5,2)$	$\frac{75-\sqrt{5177}}{150}$
	$(1,-2,-2)$	$\frac{75-\sqrt{4857}}{150}$	8	$(4,3,-1)$	$\frac{1875-\sqrt{3103081-41216\sqrt{6}}}{3750}$
	$(0,4,1)$	$\frac{12-\sqrt{139}}{24}$		$(4,1,-3)$	$\frac{2601-\sqrt{3(1995611+11520\sqrt{2})}}{5202}$
	$(0,3,2)$	$\frac{81-\sqrt{5601}}{162}$		$(3,5,0)$	$\frac{51-\sqrt{2301}}{102}$
6	$(3,2,-1)$	$\frac{507-\sqrt{232217-7040\sqrt{3}}}{1014}$		$(3,4,-1)$	$\frac{1875-\sqrt{3103081+41216\sqrt{6}}}{3750}$
	$(3,1,-2)$	$\frac{75-\sqrt{5101-64\sqrt{6}}}{150}$		$(3,3,-2)$	$\frac{81-\sqrt{6261}}{162}$
	$(2,3,-1)$	$\frac{507-\sqrt{232217+7040\sqrt{3}}}{1014}$		$(2,5,1)$	$\frac{2523-\sqrt{3(1970291-40320\sqrt{7})}}{5046}$
	$(1,5,0)$	$\frac{117-\sqrt{13389}}{234}$		$(2,5,-1)$	$\frac{2523-\sqrt{3(1970291+40320\sqrt{7})}}{5046}$
	$(1,4,1)$	$\frac{867-\sqrt{674889}}{1734}$		$(2,-3,-3)$	$\frac{507-\sqrt{237849}}{1014}$
	$(1,-3,-2)$	$\frac{75-\sqrt{5101+64\sqrt{6}}}{150}$		$(1,7,0)$	$\frac{98}{75(225+\sqrt{50037})}$
	$(0,5,1)$	$\frac{75-\sqrt{5497}}{150}$		$(1,6,1)$	$\frac{4107-\sqrt{15926649}}{8214}$
7	$(4,0,-3)$	$\frac{12-\sqrt{123}}{24}$		$(1,5,2)$	$\frac{507-\sqrt{3(79151-960\sqrt{22})}}{1014}$
	$(3,4,0)$	$\frac{75-\sqrt{4857}}{150}$		$(1,-4,-3)$	$\frac{2601-\sqrt{3(1995611-11520\sqrt{2})}}{5202}$
	$(3,3,-1)$	$\frac{81-\sqrt{5793}}{162}$		$(1,-5,-2)$	$\frac{507-\sqrt{3(79151+960\sqrt{22})}}{1014}$
	$(3,2,-2)$	$\frac{507-\sqrt{237849}}{1014}$		$(0,7,1)$	$\frac{128}{147(147+\sqrt{21353})}$
	$(3,-1,-3)$	$\frac{75-\sqrt{4857}}{150}$		$(0,5,3)$	$\frac{75-\sqrt{4857}}{150}$

Table 4.3: Vectors of single resonances of order 3, 4, 5, 6, 7 and 8 in the spatial restricted circular three-body problem.

4.4 Analysis of the stability of L_4

We know from Subsection 2.4.2 in Chapter 2 that in the non-resonant case the set S must be zero and then the equilibrium point L_4 is Lie stable. So, in this section we study the stability and instability in the resonant cases of order 3 up to order 8, depending on the dimension of the set S. There are 21 resonance vectors whose components change of sign, in this case dim $S = 0$. There are 27 resonance vectors whose components do not change sign. In this case dim $S = 1$.

4.4.1 Dimension of set S is zero

According to Remark 2.4.2 in the case of single resonance and the components of its resonance vector change of sign, then the set S is null and we have Lie stability. This situation for the equilibrium point L_4 for resonance up to order 8 correspond to the following 21 cases:

$$(2,1,-1),(2,2,-1),(1,-2,-2),(3,2,-1),(3,1,-2),(2,3,-1),(1,-3,-2),$$
$$(4,0,-3),(3,3,-1),(3,2,-2),(3,-1,-3),(2,4,-1),(1,-4,-2),(4,3,-1),$$
$$(4,1,-3),(3,4,-1),(3,3,-2),(2,5,-1),(2,-3,-3),(1,-4,-3),(1,-5,-2).$$

4.4.2 Dimension of set S is one

The resonance vectors where the components do not change sign up to order 8 correspond to the 27 cases described in Table 4.3. We observe in 24 of these cases we can apply our Theorem 2.1. In fact, it is enough to consider the truncated Hamiltonian in its Lie normal form (for more details see [55]) up to order four as follows in Tables 4.4 and 4.5.

$$\mathcal{H}^4 = \mathcal{H}_2 + \mathcal{H}_4, \tag{4.6}$$

here $\mathcal{H}_3 = 0$, whose quartic part are given in Tables 4.4 and 4.5.

Vector	$\mathcal{H}_4$
$(0,2,1)$	$0.9375I_1^2 + 23.094I_2I_1 + 0.355292I_3I_1 + 7.10417I_2^2 - 0.00512821I_3^2 + 0.533333I_2I_3$
$(0,3,1)$	$0.162822I_1^2 - 2.78956I_2I_1 + 0.115446I_3I_1 + 0.119331I_2^2 - 0.00272109I_3^2 + 0.261224I_2I_3$
$(2,3,0)$	$2.00344I_1^2 + 22.2184I_2I_1 + 0.53664I_3I_1 + 7.32821I_2^2 - 0.00581395I_3^2 + 0.72111I_2I_3$
$(1,3,1)$	$4.54617I_1^2 + 33.4796I_2I_1 + 0.816327I_3I_1 + 10.8765I_2^2 - 0.00627943I_3^2 + 1.00471I_2I_3$

Table 4.4: Lie normal form of the quartic part in cases of dim $S = 1$.

Vector	$\mathcal{H}_4$
$(0,4,1)$	$0.0711043I_1^2 - 0.971707I_2I_1 + 0.0602213I_3I_1 + 0.469813I_2^2 - 0.0016197I_3^2 + 0.181406I_2I_3$
$(0,3,2)$	$37.6024I_1^2 + 181.541I_2I_1 + 2.3082I_3I_1 + 52.1181I_2^2 - 0.00672043I_3^2 + 2.5I_2I_3$
$(1,5,0)$	$0.0389875I_1^2 - 0.554882I_2I_1 + 0.0358581I_3I_1 + 0.527952I_2^2 - 0.00102413I_3^2 + 0.137514I_2I_3$
$(1,4,1)$	$0.660714I_1^2 + 38.8306I_2I_1 + 0.290344I_3I_1 + 10.8894I_2^2 - 0.00472138I_3^2 + 0.464132I_2I_3$
$(0,5,1)$	$0.0408405I_1^2 - 0.576994I_2I_1 + 0.0373683I_3I_1 + 0.525347I_2^2 - 0.00106326I_3^2 + 0.140536I_2I_3$
$(3,4,0)$	$4.54617I_1^2 + 33.4796I_2I_1 + 0.816327I_3I_1 + 10.8765I_2^2 - 0.00627943I_3^2 + 1.00471I_2I_3$
$(2,5,0)$	$0.235536I_1^2 - 5.39917I_2I_1 + 0.150292I_3I_1 - 0.465019I_2^2 - 0.00327054I_3^2 + 0.305281I_2I_3$
$(2,4,1)$	$11.2334I_1^2 + 64.6007I_2I_1 + 1.27792I_3I_1 + 19.9316I_2^2 - 0.0065587I_3^2 + 1.4685I_2I_3$
$(1,6,0)$	$0.0259851I_1^2 - 0.403762I_2I_1 + 0.0248276I_3I_1 + 0.543756I_2^2 - 0.000728863I_3^2 + 0.113498I_2I_3$
$(1,5,1)$	$0.268184I_1^2 - 7.17963I_2I_1 + 0.164276I_3I_1 - 0.882159I_2^2 - 0.00346488I_3^2 + 0.322193I_2I_3$
$(1,4,2)$	$107.331I_1^2 + 478.445I_2I_1 + 3.8589I_3I_1 + 130.867I_2^2 - 0.006772I_3^2 + 4.05109I_2I_3$
$(0,6,1)$	$0.0267988I_1^2 - 0.413085I_2I_1 + 0.0255416I_3I_1 + 0.542894I_2^2 - 0.000748487I_3^2 + 0.115179I_2I_3$
$(0,5,2)$	$0.312489I_1^2 - 10.654I_2I_1 + 0.181984I_3I_1 - 1.71478I_2^2 - 0.00369198I_3^2 + 0.343137I_2I_3$
$(3,5,0)$	$1.12805I_1^2 + 21.2646I_2I_1 + 0.393983I_3I_1 + 6.73828I_2^2 - 0.00532028I_3^2 + 0.573913I_2I_3$
$(2,5,1)$	$1.54356I_1^2 + 20.9421I_2I_1 + 0.467317I_3I_1 + 6.83583I_2^2 - 0.00560871I_3^2 + 0.649952I_2I_3$
$(1,7,0)$	$0.0186251I_1^2 - 0.319215I_2I_1 + 0.018211I_3I_1 + 0.550798I_2^2 - 0.000543557I_3^2 + 0.0967287I_2I_3$
$(1,6,1)$	$0.149222I_1^2 - 2.44105I_2I_1 + 0.108214I_3I_1 + 0.192615I_2^2 - 0.00259416I_3^2 + 0.251636I_2I_3$
$(1,5,2)$	$2.36647I_1^2 + 23.6068I_2I_1 + 0.585359I_3I_1 + 7.79951I_2^2 - 0.00592947I_3^2 + 0.770831I_2I_3$
$(0,7,1)$	$0.0190402I_1^2 - 0.324025I_2I_1 + 0.0185921I_3I_1 + 0.550438I_2^2 - 0.000554401I_3^2 + 0.0977632I_2I_3$
$(0,5,3)$	$4.54617I_1^2 + 33.4796I_2I_1 + 0.816327I_3I_1 + 10.8765I_2^2 - 0.00627943I_3^2 + 1.00471I_2I_3$

Table 4.5: Lie normal form of the quartic part in cases of dim $S = 1$.

In order to apply the Theorem 2.1, we compute the set S in each case. Taking $I \in S$ we get $\mathcal{H}_4(I)$ as in Table 4.6.

Vector	Set S	$\mathcal{H}_4(I,\phi)$	Vector	Set S	$\mathcal{H}_4(I,\phi)$
$(0,2,1)$	$\{(0,2I_3,I_3) \mid I_3>0\}$	$29.4782I_3^2$	$(0,3,1)$	$\{(0,3I_3,I_3) \mid I_3>0\}$	$1.85493I_3^2$
$(2,3,0)$	$\{(2I_2,3I_2,0) \mid I_2>0\}$	$23.0309I_2^2$	$(1,3,1)$	$\{(I_3,3I_3,I_3) \mid I_3>0\}$	$206.698I_3^2$
$(0,4,1)$	$\{(0,4I_3,I_3) \mid I_3>0\}$	$8.24101I_3^2$	$(0,3,2)$	$\{(0,3I_3,2I_3) \mid I_3>0\}$	$121.009I_3^2$
$(1,5,0)$	$\{(I_2,5I_2,0) \mid I_2>0\}$	$0.418535I_2^2$	$(1,4,1)$	$\{(I_3,4I_3,I_3) \mid I_3>0\}$	$332.355I_3^2$
$(0,5,1)$	$\{(0,5I_3,I_3) \mid I_3>0\}$	$13.8353I_3^2$	$(3,4,0)$	$\{(3I_2,4I_2,0) \mid I_2>0\}$	$38.5434I_2^2$
$(2,5,0)$	$\{(2I_2,5I_2,0) \mid I_2>0\}$	$-2.587I_2^2$	$(2,4,1)$	$\{(2I_3,4I_3,I_3) \mid I_3>0\}$	$889.069I_3^2$
$(1,6,0)$	$\{(I_2,6I_2,0) \mid I_2>0\}$	$0.477185I_2^2$	$(1,5,1)$	$\{(I_3,5I_3,I_3) \mid I_3>0\}$	$-55.9121I_3^2$

Table 4.6: Set S and $\mathcal{H}_4(I)$ with $I \in S \setminus \{0\}$.

Vector	Set S	$\mathcal{H}_4(I)$	Vector	Set S	$\mathcal{H}_4(I)$
$(1,4,2)$	$\{(I_3,4I_3,2I_3) \mid I_3>0\}$	$1038.77I_3^2$	$(0,6,1)$	$\{(0,6I_3,I_3) \mid I_3>0\}$	$20.2345I_3^2$
$(0,5,2)$	$\{(0,5I_3,2I_3) \mid I_3>0\}$	-9.86321I_3^2	$(3,5,0)$	$\{(3I_2,5I_2,0) \mid I_3>0\}$	$19.9031I_2^2$
$(2,5,1)$	$\{(2I_3,5I_3,I_3) \mid I_3>0\}$	$390.67I_3^2$	$(1,7,0)$	$\{(I_2,7I_2,0) \mid I_2>0\}$	$0.505576I_2^2$
$(1,6,1)$	$\{(I_3,6I_3,I_3) \mid I_3>0\}$	$-5.94754I_3^2$	$(1,5,2)$	$\{(I_3,5I_3,2I_3) \mid I_3>0\}$	$81.0609I_3^2$
$(0,7,1)$	$\{(0,7I_3,I_3) \mid I_3>0\}$	$27.6552I_3^2$	$(0,5,3)$	$\{(0,5I_3,3I_3) \mid I_3>0\}$	$31.8808I_3^2$

Table 4.7: Set S and $\mathcal{H}_4(I)$ with $I \in S \setminus \{0\}$.

From the last column of the Table 4.6 and 4.7 it is readily deduced that $\mathcal{H}_4(I) \neq 0$, for $I \in S\setminus\{0\}$. Thus, the origin of $\mathbb{R}^6$ is Lie stable for the Hamiltonian system associated to (4.6) by virtue of Theorem 2.1. Note that is not necessary to normalize the Hamiltonian (4.6) for terms of order greater than 4, independent of the order of the resonance.

For the remaining two cases $k_1 = (1,2,0)$, $k_2 = (1,3,0)$ we need to normalize up to the order of resonance, respectively, as follow

$$\begin{aligned}
\mathcal{H}^3 \quad &= \tfrac{1}{2}\left(1.78885I_1 - 0.894427I_2 + 2I_3\right) - 0.13155\sqrt{I_1}I_2\sin\left(\varphi_1 + 2\varphi_2\right) - \\
&1.34902\sqrt{I_1}I_2\cos\left(\varphi_1 + 2\varphi_2\right), \\
\mathcal{H}^4 \quad &= 0.948683I_1 - 0.316228I_2 + I_3 + 0.246875I_2^2 - 2.17679I_1I_2 + \\
&0.243252I_2I_3 + 0.137946I_1^2 - 0.00248139I_3^2 + 0.102009I_1I_3 - \\
&0.702366\sqrt{I_1}I_2^{3/2}\sin\left(\varphi_1 + 3\varphi_2\right) - 4.42535\sqrt{I_1}I_2^{3/2}\cos\left(\varphi_1 + 3\varphi_2\right).
\end{aligned} \tag{4.7}$$

In these cases we are going to apply our Theorem 3.2. Here we need to compute $\mathcal{H}^{|k_j|}(k_j)$:

$$\begin{aligned}
\mathcal{H}^3(k_1) \quad &= -0.2631\sin\left(\varphi_1 + 2\varphi_2\right) - 2.69804\cos\left(\varphi_1 + 2\varphi_2\right), \\
\mathcal{H}^4(k_2) \quad &= -4.17054 - 3.6496\sin\left(\varphi_1 + 3\varphi_2\right) - 22.9948\cos\left(\varphi_1 + 3\varphi_2\right).
\end{aligned}$$

It is easy to verify that for each case the hypothesis of the Theorem 3.2 is fulfilled and therefore in these three situations the equilibrium point L_4 is unstable in the Liapunov sense.

4.5 Asymptotic estimates

For the cases where the equilibrium point L_4 is Lie stable studied previously we are going to apply our Theorem 2.2 of asymptotic estimates and we maintain the notation of this Theorem. We will divide the analysis according the dimension of the set S. When the dimension of set S is zero, we notice that $H_2 = \sigma_1 F_1 + \sigma_2 F_2$ with F_1, F_2, σ_1 and σ_2 are as in the following Table 4.8.

Vector	H_2	F_1	F_2	$\sigma = (\sigma_1, \sigma_2)$
$(2,1,-1)$	$\frac{4I_1}{5} + I_3 - \frac{3I_2}{5}$	$I_1 + 2I_3$	$2I_2 - I_1$	$\left(\frac{1}{2}, -\frac{3}{10}\right)$
$(2,2,-1)$	$\frac{1}{4}\left(\sqrt{7}I_1 + I_1 - \sqrt{7}I_2 + I_2 + 4I_3\right)$	$I_1 + 2I_3$	$I_2 - I_1$	$\left(\frac{1}{2}, \frac{1}{4}\left(1-\sqrt{7}\right)\right)$
$(1,-2,-2)$	$\frac{4I_1}{5} + I_3 - \frac{3I_2}{5}$	$2I_1 + I_3$	$2I_1 + I_2$	$\left(1, -\frac{3}{5}\right)$
$(3,2,-1)$	$\frac{1}{13}\left(\left(3+4\sqrt{3}\right)I_1 + \left(2-6\sqrt{3}\right)I_2 + 13I_3\right)$	$I_1 + 3I_3$	$3I_2 - 2I_1$	$\left(\frac{1}{3}, \frac{1}{39}(-2)\left(-1+3\sqrt{3}\right)\right)$
$(3,1,-2)$	$\frac{1}{10}\left(\left(6+\sqrt{6}\right)I_1 + \left(2-3\sqrt{6}\right)I_2 + 10I_3\right)$	$2I_1 + 3I_3$	$3I_2 - I_1$	$\left(\frac{1}{3}, \frac{1}{30}\left(2-3\sqrt{6}\right)\right)$
$(2,3,-1)$	$\frac{1}{13}\left(\left(2+6\sqrt{3}\right)I_1 + \left(3-4\sqrt{3}\right)I_2 + 13I_3\right)$	$I_1 + 2I_3$	$2I_2 - 3I_1$	$\left(\frac{1}{2}, \frac{1}{26}\left(3-4\sqrt{3}\right)\right)$
$(1,-3,-2)$	$\frac{1}{10}\left(\left(2+3\sqrt{6}\right)I_1 + \left(-6+\sqrt{6}\right)I_2 + 10I_3\right)$	$2I_1 + I_3$	$3I_1 + I_2$	$\left(1, \frac{1}{10}\left(-6+\sqrt{6}\right)\right)$
$(4,0,-3)$	$\frac{3I_1}{4} + I_3 - \frac{\sqrt{7}I_2}{4}$	$3I_1 + 4I_3$	I_2	$\left(\frac{1}{4}, -\frac{\sqrt{7}}{4}\right)$
$(3,3,-1)$	$\frac{1}{6}\left(\sqrt{17}I_1 + I_1 - \sqrt{17}I_2 + I_2 + 6I_3\right)$	$I_1 + 3I_3$	$I_2 - I_1$	$\left(\frac{1}{3}, \frac{1}{6}\left(1-\sqrt{17}\right)\right)$
$(3,2,-2)$	$\frac{12I_1}{13} + I_3 - \frac{5I_2}{13}$	$2I_1 + 3I_3$	$3I_2 - 2I_1$	$\left(\frac{1}{3}, -\frac{5}{39}\right)$
$(3,-1,-3)$	$I_1 + I_3$	$I_1 + I_3$	$I_1 + 3I_2$	$(1,0)$
$(2,4,-1)$	$\frac{1}{10}\left(2\sqrt{19}I_1 + I_1 - \left(-2+\sqrt{19}\right)I_2 + 10I_3\right)$	$I_1 + 2I_3$	$I_2 - 2I_1$	$\left(\frac{1}{2}, \frac{1}{10}\left(2-\sqrt{19}\right)\right)$
$(1,-4,-2)$	$\frac{1}{17}\left(\left(2+4\sqrt{13}\right)I_1 + \left(-8+\sqrt{13}\right)I_2 + 17I_3\right)$	$2I_1 + I_3$	$4I_1 + I_2$	$\left(1, \frac{1}{17}\left(-8+\sqrt{13}\right)\right)$
$(4,3,-1)$	$\frac{14\sqrt{2}I_1+92\sqrt{3}I_1+48\sqrt{2}I_2-131\sqrt{3}I_2+200\sqrt{2}I_3-25\sqrt{3}I_3}{25\sqrt{131-16\sqrt{6}}}$	$I_1 + 4I_3$	$4I_2 - 3I_1$	$\left(\frac{1}{4}, \frac{1}{100}\left(3-8\sqrt{6}\right)\right)$
$(4,1,-3)$	$\frac{2}{17}\left(6+\sqrt{2}\right)I_1 + \frac{1}{17}\left(3-8\sqrt{2}\right)I_2 + I_3$	$3I_1 + 4I_3$	$4I_2 - I_1$	$\left(\frac{1}{4}, \frac{1}{68}\left(3-8\sqrt{2}\right)\right)$
$(3,4,-1)$	$\frac{1}{25}\left(\left(3+8\sqrt{6}\right)I_1 + \left(4-6\sqrt{6}\right)I_2 + 25I_3\right)$	$I_1 + 3I_3$	$3I_2 - 4I_1$	$\left(\frac{1}{3}, \frac{1}{75}(-2)\left(-2+3\sqrt{6}\right)\right)$
$(3,3,-2)$	$\frac{1}{6}\left(2+\sqrt{14}\right)I_1 - \frac{1}{6}\left(-2+\sqrt{14}\right)I_2 + I_3$	$2I_1 + 3I_3$	$I_2 - I_1$	$\left(\frac{1}{3}, \frac{1}{6}\left(2-\sqrt{14}\right)\right)$
$(2,5,-1)$	$\frac{1}{29}\left(2\left(1+5\sqrt{7}\right)I_1 + \left(5-4\sqrt{7}\right)I_2 + 29I_3\right)$	$I_1 + 2I_3$	$2I_2 - 5I_1$	$\left(\frac{1}{2}, \frac{1}{58}\left(5-4\sqrt{7}\right)\right)$
$(2,-3,-3)$	$\frac{12I_1}{13} + I_3 - \frac{5I_2}{13}$	$3I_1 + 2I_3$	$3I_1 + 2I_2$	$\left(\frac{1}{2}, -\frac{5}{26}\right)$
$(1,-4,-3)$	$\frac{1}{17}\left(\left(3+8\sqrt{2}\right)I_1 + 2\left(-6+\sqrt{2}\right)I_2 + 17I_3\right)$	$3I_1 + I_3$	$4I_1 + I_2$	$\left(1, \frac{2}{17}\left(-6+\sqrt{2}\right)\right)$
$(1,-5,-2)$	$\frac{1}{26}\left(\left(2+5\sqrt{22}\right)I_1 + \left(-10+\sqrt{22}\right)I_2 + 26I_3\right)$	$2I_1 + I_3$	$5I_1 + I_2$	$\left(1, \frac{1}{26}\left(-10+\sqrt{22}\right)\right)$

Table 4.8: Quadratic part, first integrals and vector σ in cases of dim $S = 0$.

The frequency vectors (σ_1, σ_2) are Diophantine vector except in the cases of resonance: $(2,1,-1), (1,-2,-2), (3,2,-2), (3,-1,-3), (2,-3,-3)$ of the Table 4.8. According to Remark 3.1 the estimates of Theorem 2.2 apply with $j = 2$, $\nu = 2$ and we obtain

$$|I(t)| < a\,\varepsilon^2 \quad \text{for all} \;\; t \;\; \text{with} \quad 0 \leq t \leq T = C \exp\left(\frac{K}{\varepsilon^{1/3}}\right).$$

Finally, when the dimension of set S is one, we notice that $H_2 = \sigma_1 F_1 + \sigma_2 F_2$ with F_1, F_2, σ_1 and σ_2 are as in the following Table 4.9.

Vector	H_2	F_1	F_2	$\sigma=(\sigma_1,\sigma_2)$
$(0,2,1)$	$\frac{\sqrt{3}I_1}{2}-\frac{I_2}{2}+I_3$	I_2-2I_3	I_1	$\left(\frac{1}{2},\frac{\sqrt{3}}{2}\right)$
$(0,3,1)$	$\frac{2\sqrt{2}I_1}{3}-\frac{I_2}{3}+I_3$	I_2-3I_3	I_1	$\left(\frac{1}{3},\frac{2\sqrt{2}}{3}\right)$
$(2,3,0)$	$\frac{3I_1}{\sqrt{13}}-\frac{2I_2}{\sqrt{13}}+I_3$	I_3	$3I_1-2I_2$	$\left(1,-\frac{1}{\sqrt{13}}\right)$
$(1,3,1)$	$\frac{4I_1}{5}-\frac{3I_2}{5}+I_3$	I_1-I_3	$3I_1-I_2$	$(1,-\frac{3}{5})$
$(0,4,1)$	$\frac{\sqrt{15}I_1}{4}-\frac{I_2}{4}+I_3$	I_2-4I_3	I_1	$\left(\frac{1}{4},\frac{\sqrt{15}}{4}\right)$
$(0,3,2)$	$\frac{\sqrt{5}I_1}{3}-\frac{2I_2}{3}+I_3$	$2I_2-3I_3$	I_1	$\left(\frac{1}{3},\frac{\sqrt{5}}{3}\right)$
$(1,5,0)$	$\frac{5I_1}{\sqrt{26}}-\frac{I_2}{\sqrt{26}}+I_3$	I_3	$5I_1-I_2$	$\left(1,-\frac{1}{\sqrt{26}}\right)$
$(1,4,1)$	$\frac{15I_1}{17}-\frac{8I_2}{17}+I_3$	I_1-I_3	$4I_1-I_2$	$(1,-\frac{8}{17})$
$(0,5,1)$	$\frac{2\sqrt{6}I_1}{5}-\frac{I_2}{5}+I_3$	I_2-5I_3	I_1	$\left(\frac{1}{5},\frac{2\sqrt{6}}{5}\right)$
$(3,4,0)$	$\frac{4I_1}{5}-\frac{3I_2}{5}+I_3$	I_3	$4I_1-3I_2$	$(1,-\frac{1}{5})$
$(2,5,0)$	$\frac{5I_1}{\sqrt{29}}-\frac{2I_2}{\sqrt{29}}+I_3$	I_3	$5I_1-2I_2$	$\left(1,-\frac{1}{\sqrt{29}}\right)$
$(2,4,1)$	$\frac{1}{10}\left(2\sqrt{19}-1\right)I_1-\frac{1}{10}\left(2+\sqrt{19}\right)I_2+I_3$	I_1-2I_3	$2I_1-I_2$	$\left(\frac{1}{2},\frac{1}{10}\left(-2-\sqrt{19}\right)\right)$
$(1,6,0)$	$\frac{6I_1}{\sqrt{37}}-\frac{I_2}{\sqrt{37}}+I_3$	I_3	$6I_1-I_2$	$\left(1,-\frac{1}{\sqrt{37}}\right)$
$(1,5,1)$	$\frac{12I_1}{13}-\frac{5I_2}{13}+I_3$	I_1-I_3	$5I_1-I_2$	$(1,-\frac{5}{13})$
$(1,4,2)$	$\frac{1}{17}\left(\left(4\sqrt{13}-2\right)I_1-\left(8+\sqrt{13}\right)I_2+17I_3\right)$	$2I_1-I_3$	$4I_1-I_2$	$\left(1,\frac{1}{17}\left(-8-\sqrt{13}\right)\right)$
$(0,6,1)$	$\frac{\sqrt{35}I_1}{6}-\frac{I_2}{6}+I_3$	I_2-6I_3	I_1	$\left(\frac{1}{6},\frac{\sqrt{35}}{6}\right)$
$(0,5,2)$	$\frac{\sqrt{21}I_1}{5}-\frac{2I_2}{5}+I_3$	$2I_2-5I_3$	I_1	$\left(\frac{1}{5},\frac{\sqrt{21}}{5}\right)$
$(3,5,0)$	$\frac{5I_1}{\sqrt{34}}-\frac{3I_2}{\sqrt{34}}+I_3$	I_3	$5I_1-3I_2$	$\left(1,-\frac{1}{\sqrt{34}}\right)$
$(2,5,1)$	$\frac{1}{29}\left(2\left(5\sqrt{7}-1\right)I_1-\left(5+4\sqrt{7}\right)I_2+29I_3\right)$	I_1-2I_3	$5I_1-2I_2$	$\left(\frac{1}{2},\frac{1}{58}\left(-5-4\sqrt{7}\right)\right)$
$(1,7,0)$	$\frac{7I_1}{5\sqrt{2}}+I_3-\frac{I_2}{5\sqrt{2}}$	I_3	$7I_1-I_2$	$\left(1,-\frac{1}{5\sqrt{2}}\right)$
$(1,6,1)$	$\frac{35I_1}{37}-\frac{12I_2}{37}+I_3$	I_1-I_3	$6I_1-I_2$	$(1,-\frac{12}{37})$
$(1,5,2)$	$\frac{1}{26}\left(\left(5\sqrt{22}-2\right)I_1-\left(10+\sqrt{22}\right)I_2+26I_3\right)$	$2I_1-I_3$	$5I_1-I_2$	$\left(1,\frac{1}{26}\left(-10-\sqrt{22}\right)\right)$
$(0,7,1)$	$\frac{4\sqrt{3}I_1}{7}-\frac{I_2}{7}+I_3$	I_2-7I_3	I_1	$\left(\frac{1}{7},\frac{4\sqrt{3}}{7}\right)$
$(0,5,3)$	$\frac{4I_1}{5}-\frac{3I_2}{5}+I_3$	$3I_2-5I_3$	I_1	$(\frac{1}{5},\frac{4}{5})$

Table 4.9: Quadratic part, first integrals and vector σ in cases of dim $S=1$.

The frequency vectors (σ_1,σ_2) are Diophantine vector except in the cases of resonance: $(1,3,1),(1,4,1),(3,4,0),(1,5,1),(1,6,1),(0,5,3)$ of the Table 4.9. Thus, the estimates of Theorem 2.2 hold with $j=4$, $\nu=2$ and we obtain

$$|I(t)|<a\,\varepsilon \quad \text{for all } t \text{ with } \quad 0\le t\le T=C\exp\left(\frac{K}{\varepsilon^{1/3}}\right).$$

Chapter 5

Application to the spatial satellite problem

In this chapter we are going to apply our main results of stability and instability to the spatial satellite problem and also we obtain results for Nekhoroshev stability. First, determine the equilibrium solutions, then characterize the resonance type according to the parameters and classify the resonances according to the dimension of the set S. Next, we normalize the quadratic part and determine the Lie normal form of the Hamiltonian to study the nonlinear stability. Specifically, we study stability using the result of Chapter 2 and instability using the results of Chapter 3. Finally, we obtain estimates of the Nekhoroshev type for an elliptic equilibrium points, we characterize the regions of quasi-convexity and directional quasi-convexity.

5.1 Statement of the problem and characterization of the resonance curves

As an application of our main results, we consider the motion of a satellite with respect to its center of mass in a central gravitational field. The orbit of the center is circular and the satellite has unequal principal central moments of inertia. See [52] and references therein for more details. In order to set up the Hamiltonian function associated to this problem we follow the notation in [52]. Let $Ouvw$ be a coordinate system whose origin coincides with the center of mass of the satellite and whose axes are directed along the principal central axes of the ellipsoid of inertia of the satellite. Its position relative to the orbital coordinate system $OUVW$ (the OU axis is directed along the radius vector of the center of mass, OV along the transversal, and OW along the normal to the plane of the orbit) is specified by means of the Euler angles x, y and z. Let a, b, c be the principal moments of inertia of the satellite, see Figure 5.1.

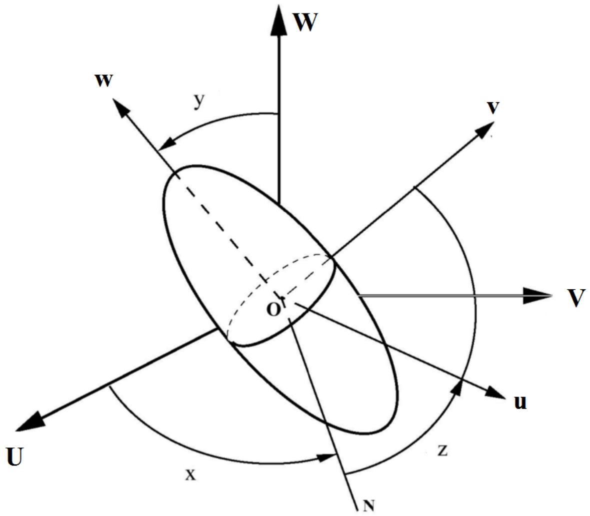


Figure 5.1: Representation of the coordinates of the satellite problem.

According to [52] the relative motion of the satellite can be described by the canonical system of differential equations associated to the autonomous Hamiltonian function with three degrees of freedom

$$
\begin{aligned}
H \; &= \tfrac{1}{2}Y^2\left(\tfrac{A}{C}+\cot^2(x)\left(A\cos^2(y)+\sin^2(y)\right)\right)-\tfrac{1}{2}(1-A)XY\cot(x)\sin(2y)+\\
&\tfrac{3}{2}\left(\tfrac{(C-1)(\sin(x)\sin(z))^2}{A}+\tfrac{(A-1)(\cos(y)\cos(z)-\cos(x)\sin(y)\sin(z))^2}{A}\right)+\\
&\tfrac{(1-A)XZ\sin(2y)}{2\sin(x)}-\tfrac{YZ\cos(x)\left(A\cos^2(y)+\sin^2(y)\right)}{\sin^2(x)}+\\
&\tfrac{Z^2\left(A\cos^2(y)+\sin^2(y)\right)}{2\sin^2(x)}+\tfrac{1}{2}X^2\left(A\sin^2(y)+\cos^2(y)\right)-Z,
\end{aligned}
$$

where $A=a/b$, $C=c/b$ are positive parameters, with the restriction

$$C+1\geq A,\quad A+1\geq C,\quad A+C\geq 1.$$

It is easily verified that this problem possesses 32 equilibria, namely:

$$
\begin{array}{llll}
\left(-\frac{\pi}{2},-\frac{\pi}{2},0,0,0,1\right), & \left(-\frac{\pi}{2},-\frac{\pi}{2},-\frac{\pi}{2},0,0,1\right), & \left(-\frac{\pi}{2},-\frac{\pi}{2},\frac{\pi}{2},0,0,1\right), & \left(-\frac{\pi}{2},-\frac{\pi}{2},\pi,0,0,1\right),\\
\left(-\frac{\pi}{2},\frac{\pi}{2},0,0,0,1\right), & \left(-\frac{\pi}{2},\frac{\pi}{2},-\frac{\pi}{2},0,0,1\right), & \left(-\frac{\pi}{2},\frac{\pi}{2},\frac{\pi}{2},0,0,1\right), & \left(-\frac{\pi}{2},\frac{\pi}{2},\pi,0,0,1\right),\\
\left(\frac{\pi}{2},-\frac{\pi}{2},0,0,0,1\right), & \left(\frac{\pi}{2},-\frac{\pi}{2},-\frac{\pi}{2},0,0,1\right), & \left(\frac{\pi}{2},-\frac{\pi}{2},\frac{\pi}{2},0,0,1\right), & \left(\frac{\pi}{2},-\frac{\pi}{2},\pi,0,0,1\right),\\
\left(\frac{\pi}{2},\frac{\pi}{2},0,0,0,1\right), & \left(\frac{\pi}{2},\frac{\pi}{2},-\frac{\pi}{2},0,0,1\right), & \left(\frac{\pi}{2},\frac{\pi}{2},\frac{\pi}{2},0,0,1\right), & \left(\frac{\pi}{2},\frac{\pi}{2},\pi,0,0,1\right),
\end{array}
$$

$$\left(-\tfrac{\pi}{2},0,0,0,0,\tfrac{1}{A}\right), \quad \left(-\tfrac{\pi}{2},0,-\tfrac{\pi}{2},0,0,\tfrac{1}{A}\right), \quad \left(-\tfrac{\pi}{2},0,\tfrac{\pi}{2},0,0,\tfrac{1}{A}\right), \quad \left(-\tfrac{\pi}{2},0,\pi,0,0,\tfrac{1}{A}\right),$$
$$\left(-\tfrac{\pi}{2},\pi,0,0,0,\tfrac{1}{A}\right), \quad \left(-\tfrac{\pi}{2},\pi,-\tfrac{\pi}{2},0,0,\tfrac{1}{A}\right), \quad \left(-\tfrac{\pi}{2},\pi,\tfrac{\pi}{2},0,0,\tfrac{1}{A}\right), \quad \left(-\tfrac{\pi}{2},\pi,\pi,0,0,\tfrac{1}{A}\right),$$
$$\left(\tfrac{\pi}{2},0,0,0,0,\tfrac{1}{A}\right), \quad \left(\tfrac{\pi}{2},0,-\tfrac{\pi}{2},0,0,\tfrac{1}{A}\right), \quad \left(\tfrac{\pi}{2},0,\tfrac{\pi}{2},0,0,\tfrac{1}{A}\right), \quad \left(\tfrac{\pi}{2},0,\pi,0,0,\tfrac{1}{A}\right),$$
$$\left(\tfrac{\pi}{2},\pi,0,0,0,\tfrac{1}{A}\right), \quad \left(\tfrac{\pi}{2},\pi,-\tfrac{\pi}{2},0,0,\tfrac{1}{A}\right), \quad \left(\tfrac{\pi}{2},\pi,\tfrac{\pi}{2},0,0,\tfrac{1}{A}\right), \quad \left(\tfrac{\pi}{2},\pi,\pi,0,0,\tfrac{1}{A}\right).$$

Due to the symmetries of the problem, which are:

$$\begin{aligned}
S_1 &: (x,y,z,X,Y,Z) \to (-x,y,z,-X,Y,Z),\\
S_2 &: (x,y,z,X,Y,Z) \to (x,-y,z,-X,-Y,Z),\\
S_3 &: (x,y,z,X,Y,Z) \to (x,y+\pi,z,X,Y,Z),\\
S_4 &: (x,y,z,X,Y,Z) \to (x,y,z+\pi,X,Y,Z),
\end{aligned}$$

the study of the stability of them can be reduced to 6, namely,

$$\begin{aligned}
P_1 &= \left(\tfrac{\pi}{2},-\tfrac{\pi}{2},0,0,0,1\right), & P_2 &= \left(\tfrac{\pi}{2},-\tfrac{\pi}{2},-\tfrac{\pi}{2},0,0,1\right), & P_3 &= \left(\tfrac{\pi}{2},0,0,0,0,\tfrac{1}{A}\right),\\
P_4 &= \left(\tfrac{\pi}{2},0,-\tfrac{\pi}{2},0,0,\tfrac{1}{A}\right), & P_5 &= \left(\tfrac{\pi}{2},-\tfrac{\pi}{2},\tfrac{\pi}{2},0,0,1\right), & P_6 &= \left(\tfrac{\pi}{2},0,\tfrac{\pi}{2},0,0,\tfrac{1}{A}\right).
\end{aligned}$$

Note that three of them depends on the parameter A.

Analysis of the point P_1

In order to apply our main result on single resonance, we are going to study only the point P_1, because the point P_3 was studied in [52]. First, we observe that the linearization matrix associated to this equilibrium is

$$B = \begin{pmatrix}
0 & A-1 & 0 & A & 0 & 0\\
1 & 0 & 0 & 0 & \frac{A}{C} & 0\\
0 & 0 & 0 & 0 & 0 & 1\\
-1 & 0 & 0 & 0 & -1 & 0\\
0 & -A-2+\frac{3}{A} & 0 & 1-A & 0 & 0\\
0 & 0 & \frac{3}{A}-\frac{3C}{A} & 0 & 0 & 0
\end{pmatrix},$$

whose characteristic polynomial is

$$\begin{aligned}
p_B(r) &= r^6 + \tfrac{A^3-A^2C+2A^2+2AC-3A+3C^2-3C}{AC}r^4 + \tfrac{12(A-1)(C-1)(A-C)}{AC} +\\
&\tfrac{4A^3-A^2C-7A^2-3AC^2+13AC-6A+6C^2-15C+9}{AC}r^2.
\end{aligned}$$

Applying the Sturm method to determine real, distinct and negative solutions of the cubic polynomial $p_B(t)$ with $t=r^2$ and thus the degree polynomial 6, $p_B(r)$, will have pure and distinct imaginary roots. After some manipulation we verify that in region I which is delimited by the straight lines $C = 1$, $A - 1 = C$, $A = C$ while the region II which is delimited by the straight lines $A = 1$, $C = 1$, $A+1 = C$ and the curve $\alpha_1 = 0$, that comes from the last term in Sturm sequence, which is $\frac{9\alpha_1\alpha_2^2}{4\alpha_3^2}$, with

$$\begin{aligned}
\alpha_1 = {} & \left(A^2 + 2A - 3\right)^2 + (A(A+12) - 12)C^2 - 2(A-1)(A(A+9) - 6)C, \\
\alpha_2 = {} & -4A^4 + A^3(7C+1) + A^2((5-3C)C - 6) + 3A(C-1)(2C-3) - 9(C-1)^2C, \\
\alpha_3 = {} & (A-1)^2A^2(A+3)^2 + 3((A-2)A - 6)C^3 + (A(A(A(A+5) - 17) + 15) + 9) \\
& C^2 - A(A(A(2A(A+6) - 29) + 6) + 9)C + 9C^4.
\end{aligned}$$

See Figure 5.3. The pure imaginary eigenvalues are $\lambda_1 = \pm i\omega_1$, $\lambda_2 = \pm i\omega_2$,

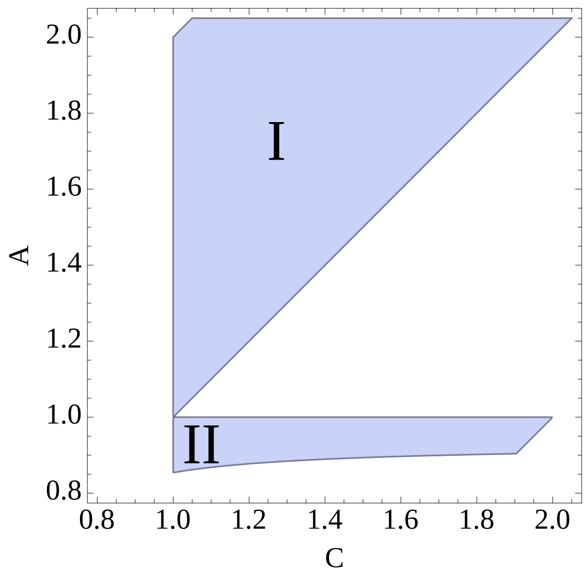


Figure 5.2: Regions I and II of existence of pure imaginary eigenvalues.

$\lambda_3 = \pm i\omega_3$, whose frequencies are

$$\omega_1 = \sqrt{\frac{\beta + \sqrt{\alpha}}{2C}}, \quad \omega_2 = \sqrt{\frac{\beta - \sqrt{\alpha}}{2C}}, \quad \omega_3 = \sqrt{\frac{3(C-1)}{A}}$$

with $\alpha = A^4 - 2A^3C + 4A^3 + A^2C^2 - 16A^2C - 2A^2 + 12AC^2 + 30AC - 12A - 12C^2 - 12C + 9$, $\beta = A^2 - AC + 2A + 2C - 3$. Using the Markeev procedure for the normalization of the quadratic part (see [51]), we determine that the region where the quadratic part is of indefinite sign is the region II in Figure 5.3, and here the quadratic part assumes the form

$$H_2 = \omega_1 I_1 - \omega_2 I_2 + \omega_3 I_3.$$

Analysis of the point P_3

In order to apply our main result on multiple resonance, estimates of time exponential and the theory of Nekhoroshev, we are going to study only the point P_3. The physical importance of this equilibrium point is described in [52] and references therein. We move the equilibrium solution P_3 at the origin by means of the following change of coordinates $x = \varepsilon x_1 + \frac{\pi}{2}$, $y = \varepsilon y_1$, $z = \varepsilon z_1$, $X = \varepsilon X_1$, $Y = \varepsilon Y_1$, $Z = \frac{1}{A} + \varepsilon Z_1$ and we expand the Hamiltonian function (5.1) in Taylor series in a neighborhood of the origin, so the Hamiltonians H_j with $j = 2, 3, 4$ are

$$
\begin{aligned}
H_2 &= \tfrac{y_1^2}{2A^2} + \tfrac{AY_1^2}{2C} + \tfrac{3Cz_1^2}{2A} + \tfrac{x_1^2}{2A} + \tfrac{X_1 y_1}{A} + \tfrac{y_1^2}{A} + \tfrac{AZ_1^2}{2} + x_1 Y_1 - X_1 y_1 + \tfrac{X_1^2}{2} - \tfrac{3y_1^2}{2} - \tfrac{3z_1^2}{2},\\
H_3 &= -\tfrac{3x_1 y_1 z_1}{A} + Ax_1 Y_1 Z_1 - AX_1 y_1 Z_1 + \tfrac{y_1^2 Z_1}{A} + 3x_1 y_1 z_1 + x_1^2 Z_1 + X_1 y_1 Z_1 - y_1^2 Z_1,\\
H_4 &= \tfrac{x_1^2 y_1^2}{2A^2} - \tfrac{y_1^4}{6A^2} - \tfrac{3Cx_1^2 z_1^2}{2A} - \tfrac{Cz_1^4}{2A} - Ax_1 X_1 y_1 Y_1 + \tfrac{x_1^2 X_1 y_1}{2A} + \tfrac{x_1 y_1^2 Y_1}{A} - \tfrac{x_1^2 y_1^2}{2A} +\\
&\quad \tfrac{1}{2}Ax_1^2 Y_1^2 + \tfrac{3x_1^2 z_1^2}{2A} + \tfrac{1}{2}Ax_1^2 Z_1^2 + \tfrac{x_1^4}{3A} + \tfrac{1}{2}AX_1^2 y_1^2 - \tfrac{2X_1 y_1^3}{3A} - \tfrac{3y_1^2 z_1^2}{2A} - \tfrac{1}{2}Ay_1^2 Z_1^2 - \tfrac{y_1^4}{3A} +\\
&\quad x_1 X_1 y_1 Y_1 - \tfrac{1}{2}x_1^2 X_1 y_1 - x_1 y_1^2 Y_1 + \tfrac{5}{6}x_1^3 Y_1 + \tfrac{2}{3}X_1 y_1^3 - \tfrac{1}{2}X_1^2 y_1^2 + \tfrac{3}{2}y_1^2 z_1^2 + \tfrac{1}{2}y_1^2 Z_1^2 +\\
&\quad \tfrac{y_1^4}{2} + \tfrac{z_1^4}{2}.
\end{aligned}
$$

First, we observe that the linearization matrix associated to this equilibrium is

$$
B = \begin{pmatrix}
0 & \frac{1}{A} - 1 & 0 & 1 & 0 & 0\\
1 & 0 & 0 & 0 & \frac{A}{C} & 0\\
0 & 0 & 0 & 0 & 0 & A\\
-\frac{1}{A} & 0 & 0 & 0 & -1 & 0\\
0 & -\frac{1}{A^2} + 3 - \frac{2}{A} & 0 & 1 - \frac{1}{A} & 0 & 0\\
0 & 0 & 3 - \frac{3C}{A} & 0 & 0 & 0
\end{pmatrix},
$$

whose characteristic polynomial is

$$
\begin{aligned}
p_B(r) &= \tfrac{9A^3 - 15A^2C - 6A^2 + 6AC^2 + 13AC - 7A - 3C^2 - C + 4}{AC} r^2\\
&\quad - \tfrac{3A^2C + 3A^2 - 3AC^2 - 2AC - 2A + C - 1}{AC} r^4 + - \tfrac{12(A-1)(C-1)(A-C)}{AC} + r^6.
\end{aligned}
$$

We apply the Sturm method to determine real, distinct and negative solutions of the cubic polynomial $p_B(t)$ with $t = r^2$ and thus polynomial $p_B(r)$, will have degree 6 in r, whose roots will be imaginary pure and distinct. After some manipulation we verify that in the regions I (I is delimited by the straight lines $A = C$, $C = 1$ and $A = 0$), and II (II is delimited by the straight lines $A = C$, $A = 1$ and the curve $\alpha_1 = 0$, that comes from the last term in Sturm sequence, which is $\frac{9\alpha_1\alpha_2^2}{4\alpha_3^2}$,

with

$$\begin{aligned}
\alpha_1 &= -2(A-1)\left(6A^2-9A-1\right)C + (1-12(A-1)A)C^2 + (A(2-3A)+1)^2\\
\alpha_2 &= 3\left(6A^2+2A-1\right)C^2 - 9AC^3 + A(5-3A(3A+5))C + A(1-3A)^2 + 7C - 4\\
\alpha_3 &= 9A^4((C-1)C+1) - 3A^3(C(C(6C-5)+2)+4)+\\
& A^2\left(C\left(C\left(9C^2-6C-17\right)+29\right)-2\right) + A(C-1)(C(3C+8)-4) + (C-1)^2,
\end{aligned}$$

the eigenvalues are pure imaginary. See Figure 5.3.

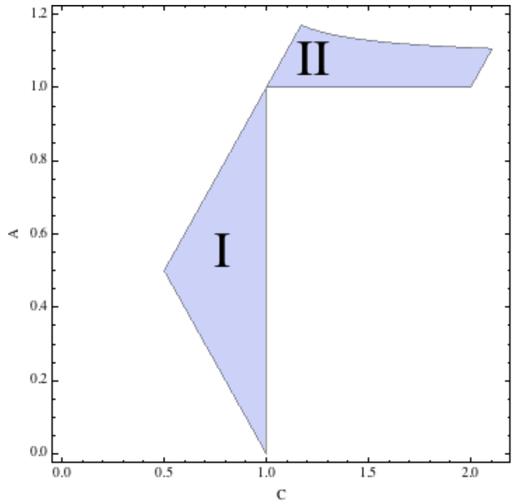


Figure 5.3: Regions I and II of existence of pure imaginary eigenvalues.

The regions I and II above described coincide with Fig. 1 in [52]. Moreover, the pure imaginary eigenvalues are $\lambda_1 = \pm i\omega_1$, $\lambda_2 = \pm i\omega_2$, $\lambda_3 = \pm i\omega_3$, whose frequencies are $\omega_1 = \frac{\sqrt{C(\alpha+\beta)}}{\sqrt{2}AC}$, $\omega_2 = \frac{\sqrt{-C(\alpha-\beta)}}{\sqrt{2}AC}$, $\omega_3 = \sqrt{3(C-A)}$ with $\alpha = \sqrt{-2(A-1)\left(6A^2-9A-1\right)C + (1-12(A-1)A)C^2 + (A(2-3A)+1)^2}$, and $\beta = A(-3A+2C+2) - C + 1$.

Using Markeev's procedure for the normalization of the quadratic part (see [51]), we determine that the region where the quadratic part is of indefinite sign is the region II in Figure 5.3, and here the quadratic part assumes the form

$$H_2 = \omega_1 I_1 - \omega_2 I_2 + \omega_3 I_3.$$

We are going to analyze the curves of resonances of order $3, 4, \ldots, 8$ in the region II, i.e., $\omega_1 k_1 - \omega_2 k_2 + \omega_3 k_3 = 0, \quad |k| = |k_1| + |k_2| + |k_3| = j$. For each $j = 3, \ldots, 8$, we compute that in the region II there are exactly 176 distinct single resonance of order j, that are shown in the Table 5.1.

n^0	Order 3	n^0	Order 4	n^0	Order 5	n^0	Order 6	n^0	Order 7	n^0	Order 8
1	$(0,1,2)$	1	$(0,1,3)$	1	$(0,1,4)$	1	$(0,1,5)$	1	$(0,1,6)$	1	$(0,1,7)$
2	$(0,2,1)$	2	$(0,3,1)$	2	$(0,2,3)$	2	$(0,5,1)$	2	$(0,2,5)$	2	$(0,3,5)$
3	$(1,-1,-1)$	3	$(1,-2,-1)$	3	$(0,3,2)$	3	$(1,-4,-1)$	3	$(0,3,4)$	3	$(0,5,3)$
4	$(1,0,-2)$	4	$(1,-1,-2)$	4	$(0,4,1)$	4	$(1,-3,-2)$	4	$(0,4,3)$	4	$(0,7,1)$
5	$(1,1,-1)$	5	$(1,0,-3)$	5	$(1,-3,-1)$	5	$(1,-2,-3)$	5	$(0,5,2)$	5	$(1,-6,-1)$
6	$(1,2,0)$	6	$(1,1,-2)$	6	$(1,-2,-2)$	6	$(1,-1,-4)$	6	$(0,6,1)$	6	$(1,-5,-2)$
7	$(2,0,-1)$	7	$(1,2,-1)$	7	$(1,-1,-3)$	7	$(1,0,-5)$	7	$(1,-5,-1)$	7	$(1,-4,-3)$
		8	$(1,2,1)$	8	$(1,0,-4)$	8	$(1,1,-4)$	8	$(1,-4,-2)$	8	$(1,-3,-4)$
		9	$(1,3,0)$	9	$(1,1,-3)$	9	$(1,2,-3)$	9	$(1,-3,-3)$	9	$(1,-2,-5)$
		10	$(2,-1,-1)$	10	$(1,2,2)$	10	$(1,2,3)$	10	$(1,-2,-4)$	10	$(1,-1,-6)$
		11	$(2,1,-1)$	11	$(1,3,-1)$	11	$(1,3,-2)$	11	$(1,-1,-5)$	11	$(1,0,-7)$
		12	$(3,0,-1)$	12	$(1,3,1)$	12	$(1,3,2)$	12	$(1,0,-6)$	12	$(1,1,-6)$
		13	$(1,2,-2)$	13	$(1,4,0)$	13	$(1,4,-1)$	13	$(1,1,-5)$	13	$(1,2,-5)$
				14	$(2,-2,-1)$	14	$(1,4,1)$	14	$(1,2,-4)$	14	$(1,2,5)$
				15	$(2,-1,-2)$	15	$(1,5,0)$	15	$(1,2,4)$	15	$(1,3,-4)$
				16	$(2,0,-3)$	16	$(2,-3,-1)$	16	$(1,3,-3)$	16	$(1,3,4)$
				17	$(2,1,-2)$	17	$(2,-1,-3)$	17	$(1,3,3)$	17	$(1,4,-3)$
				18	$(2,2,-1)$	18	$(2,1,-3)$	18	$(1,4,-2)$	18	$(1,4,3)$
				19	$(2,3,0)$	19	$(2,3,-1)$	19	$(1,4,2)$	19	$(1,5,-2)$
				20	$(3,-1,-1)$	20	$(2,3,1)$	20	$(1,5,-1)$	20	$(1,5,2)$
				21	$(3,0,-2)$	21	$(3,-2,-1)$	21	$(1,5,1)$	21	$(1,6,-1)$
				22	$(3,1,-1)$	22	$(3,-1,-2)$	22	$(1,6,0)$	22	$(1,6,1)$
				23	$(4,0,-1)$	23	$(3,1,-2)$	23	$(2,-4,-1)$	23	$(1,7,0)$
				24	$(1,2,-2)$	24	$(3,2,-1)$	24	$(2,-3,-2)$	24	$(2,-5,-1)$
						25	$(4,-1,-1)$	25	$(2,-2,-3)$	25	$(2,-3,-3)$
						26	$(4,1,-1)$	26	$(2,-1,-4)$	26	$(2,-1,-5)$
						27	$(5,0,-1)$	27	$(2,0,-5)$	27	$(2,1,-5)$
								28	$(2,1,-4)$	28	$(2,3,-3)$
								29	$(2,2,-3)$	29	$(2,3,3)$
								30	$(2,3,-2)$	30	$(2,5,-1)$
								31	$(2,3,2)$	31	$(2,5,1)$
								32	$(2,4,-1)$	32	$(3,-4,-1)$
								33	$(2,4,1)$	33	$(3,-3,-2)$
								34	$(2,5,0)$	34	$(3,-2,-3)$
								35	$(3,-3,-1)$	35	$(3,-1,-4)$
								36	$(3,-2,-2)$	36	$(3,0,-5)$
								37	$(3,-1,-3)$	37	$(3,1,-4)$
								38	$(3,0,-4)$	38	$(3,2,-3)$
								39	$(3,1,-3)$	39	$(3,3,-2)$
								40	$(3,2,-2)$	40	$(3,4,-1)$
								41	$(3,3,-1)$	41	$(3,4,1)$
								42	$(3,4,0)$	42	$(3,5,0)$
								43	$(4,-2,-1)$	43	$(4,-3,-1)$
								44	$(4,-1,-2)$	44	$(4,-1,-3)$
								45	$(4,0,-3)$	45	$(4,1,-3)$
								46	$(4,1,-2)$	46	$(4,3,-1)$
								47	$(4,2,-1)$	47	$(5,-2,-1)$
								48	$(5,-1,-1)$	48	$(5,-1,-2)$
								49	$(5,0,-2)$	49	$(5,0,-3)$
								50	$(5,1,-1)$	50	$(5,1,-2)$
								51	$(6,0,-1)$	51	$(5,2,-1)$
										52	$(6,-1,-1)$
										53	$(6,1,-1)$
										54	$(7,0,-1)$

Table 5.1: Vectors of single resonance of order $3, 4, 5, 6, 7$ and 8 in the spatial satellite problem.

Next, we classify the resonances according to the dimension of the set S. The dimension of the set S can be 0, 1 or 2.

1) dim $S = 0$. There are 124 single resonances (see Table 5.2 and 5.3)

2) dim $S = 1$. There are 52 single resonances (see Table 5.4)

3) dim $S = 2$

   3.1) Same order. There are:

      3.1.1) 8 multiple resonances of order 3 (see Table 5.5)
      3.1.2) 12 multiple resonances of order 4 (see Table 5.5)
      3.1.3) 77 multiple resonances of order 5 (see Table 5.6, 5.7 and 5.8)
      3.1.4) 71 multiple resonances of order 6
      3.1.5) 337 multiple resonances of order 7
      3.1.6) 313 multiple resonances of order 8

   3.2) Different order. There are:

      3.2.1) 2 multiple resonances of order 3 and 4 (see Table 5.9)
      3.2.2) 2 multiple resonances of order 3 and 5 (see Table 5.9)
      3.2.3) 2 multiple resonances of order 3 and 6 (see Table 5.9)
      3.2.4) 2 multiple resonances of order 3 and 7 (see Table 5.9)
      3.2.5) 2 multiple resonances of order 3 and 8 (see Table 5.9)
      3.2.6) 4 multiple resonances of order 4 and 5 (see Table 5.9)
      3.2.7) 4 multiple resonances of order 4 and 6 (see Table 5.9)
      3.2.8) 4 multiple resonances of order 4 and 7 (see Table 5.9)
      3.2.9) 4 multiple resonances of order 4 and 8 (see Table 5.9)

$\|k\|$	Resonances vectors				Set S
3	$(1,-1,-1)$	$(1,0,-2)$	$(1,1,-1)$	$(2,0,-1)$	$S=\{0\}$
4	$(1,-2,-1)$	$(1,-1,-2)$	$(1,0,-3)$	$(1,1,-2)$	
	$(1,2,-1)$	$(2,-1,-1)$	$(2,1,-1)$	$(3,0,-1)$	

Table 5.2: Single resonances with dim $S = 0$.

$\lvert k\rvert$	Resonances vectors				Set S
5	$(1,-3,-1)$	$(1,-2,-2)$	$(1,-1,-3)$	$(1,0,-4)$	$S=\{0\}$
	$(1,1,-3)$	$(1,2,-2)$	$(1,3,-1)$	$(2,-2,-1)$	
	$(2,-1,-2)$	$(2,0,-3)$	$(2,1,-2)$	$(2,2,-1)$	
	$(3,-1,-1)$	$(3,0,-2)$	$(3,1,-1)$	$(4,0,-1)$	
6	$(1,-4,-1)$	$(1,-3,-2)$	$(1,-2,-3)$	$(1,-1,-4)$	
	$(1,0,-5)$	$(1,1,-4)$	$(1,2,-3)$	$(1,3,-2)$	
	$(1,4,-1)$	$(2,-3,-1)$	$(2,-1,-3)$	$(2,1,-3)$	
	$(2,3,-1)$	$(3,-2,-1)$	$(3,-1,-2)$	$(3,1,-2)$	
	$(3,2,-1)$	$(4,-1,-1)$	$(4,1,-1)$	$(5,0,-1)$	
7	$(1,-5,-1)$	$(1,-4,-2)$	$(1,-3,-3)$	$(1,-2,-4)$	
	$(1,-1,-5)$	$(1,0,-6)$	$(1,1,-5)$	$(1,2,-4)$	
	$(1,3,-3)$	$(1,4,-2)$	$(1,5,-1)$	$(2,-4,-1)$	
	$(2,-3,-2)$	$(2,-2,-3)$	$(2,-1,-4)$	$(2,0,-5)$	
	$(2,1,-4)$	$(2,2,-3)$	$(2,3,-2)$	$(2,4,-1)$	
	$(3,-3,-1)$	$(3,-2,-2)$	$(3,-1,-3)$	$(3,0,-4)$	
	$(3,1,-3)$	$(3,2,-2)$	$(3,3,-1)$	$(4,-2,-1)$	
	$(4,-1,-2)$	$(4,0,-3)$	$(4,1,-2)$	$(4,2,-1)$	
	$(5,-1,-1)$	$(5,0,-2)$	$(5,1,-1)$	$(6,0,-1)$	
8	$(1,-6,-1)$	$(1,-5,-2)$	$(1,-4,-3)$	$(1,-3,-4)$	
	$(1,-2,-5)$	$(1,-1,-6)$	$(1,0,-7)$	$(1,1,-6)$	
	$(1,2,-5)$	$(1,3,-4)$	$(1,4,-3)$	$(1,5,-2)$	
	$(1,6,-1)$	$(2,-5,-1)$	$(2,-3,-3)$	$(2,-1,-5)$	
	$(2,1,-5)$	$(2,3,-3)$	$(2,5,-1)$	$(3,-4,-1)$	
	$(3,-3,-2)$	$(3,-2,-3)$	$(3,-1,-4)$	$(3,0,-5)$	
	$(3,1,-4)$	$(3,2,-3)$	$(3,3,-2)$	$(3,4,-1)$	
	$(4,-3,-1)$	$(4,-1,-3)$	$(4,1,-3)$	$(4,3,-1)$	
	$(5,-2,-1)$	$(5,-1,-2)$	$(5,0,-3)$	$(5,1,-2)$	
	$(5,2,-1)$	$(6,-1,-1)$	$(6,1,-1)$	$(7,0,-1)$	

Table 5.3: Single resonances with $\dim S = 0$.

$\|k\|$	Resonances	Set S	Resonances	Set S
3	$(0,1,2)$	$\{1/2(0,I_3,2I_3) \mid I_3 \geq 0\}$	$(0,2,1)$	$\{(0,2I_3,I_3) \mid I_3 \geq 0\}$
	$(1,2,0)$	$\{1/2(I_2,2I_2,0) \mid I_2 \geq 0\}$		
4	$(0,1,3)$	$\{1/3(0,I_3,3I_3) \mid I_3 \geq 0\}$	$(0,3,1)$	$\{(0,3I_3,I_3) \mid I_3 \geq 0\}$
	$(1,2,1)$	$\{(I_3,2I_3,I_3) \mid I_3 \geq 0\}$	$(1,3,0)$	$\{1/3(I_2,3I_2,0) \mid I_2 \geq 0\}$
5	$(0,1,4)$	$\{1/4(0,I_3,4I_3) \mid I_3 \geq 0\}$	$(0,2,3)$	$\{1/3(0,2I_3,3I_3) \mid I_3 \geq 0\}$
	$(0,3,2)$	$\{1/2(0,3I_3,2I_3) \mid I_3 \geq 0\}$	$(0,4,1)$	$\{(0,4I_3,I_3) \mid I_3 \geq 0\}$
	$(1,2,2)$	$\{1/2(I_3,2I_3,2I_3) \mid I_3 \geq 0\}$	$(1,3,1)$	$\{(I_3,3I_3,I_3) \mid I_3 \geq 0\}$
	$(1,4,0)$	$\{1/4(I_2,4I_2,0) \mid I_2 \geq 0\}$	$(2,3,0)$	$\{1/3(2I_2,3I_2,0) \mid I_2 \geq 0\}$
6	$(0,1,5)$	$\{1/5(0,I_3,5I_3) \mid I_3 \geq 0\}$	$(0,5,1)$	$\{(0,5I_3,I_3) \mid I_3 \geq 0\}$
	$(1,2,3)$	$\{1/3(I_3,2I_3,3I_3) \mid I_3 \geq 0\}$	$(1,3,2)$	$\{1/2(I_3,3I_3,2I_3) \mid I_3 \geq 0\}$
	$(1,4,1)$	$\{(I_3,4I_3,I_3) \mid I_3 \geq 0\}$	$(1,5,0)$	$\{1/5(I_2,5I_2,0) \mid I_2 \geq 0\}$
	$(2,3,1)$	$\{(2I_3,3I_3,I_3) \mid I_3 \geq 0\}$		
7	$(0,1,6)$	$\{1/6(0,I_3,6I_3) \mid I_3 \geq 0\}$	$(0,2,5)$	$\{1/5(0,2I_3,5I_3) \mid I_3 \geq 0\}$
	$(0,3,4)$	$\{1/4(0,3I_3,4I_3) \mid I_3 \geq 0\}$	$(0,4,3)$	$\{1/3(0,4I_3,3I_3) \mid I_3 \geq 0\}$
	$(0,5,2)$	$\{1/2(0,5I_3,2I_3) \mid I_3 \geq 0\}$	$(0,6,1)$	$\{(0,6I_3,I_3) \mid I_3 \geq 0\}$
	$(1,2,4)$	$\{1/4(I_3,2I_3,4I_3) \mid I_3 \geq 0\}$	$(1,3,3)$	$\{1/3(I_3,3I_3,3I_3) \mid I_3 \geq 0\}$
	$(1,4,2)$	$\{1/2(I_3,4I_3,2I_3) \mid I_3 \geq 0\}$	$(1,5,1)$	$\{(I_3,5I_3,I_3) \mid I_3 \geq 0\}$
	$(1,6,0)$	$\{1/6(I_2,6I_2,0) \mid I_2 \geq 0\}$	$(2,3,2)$	$\{1/2(2I_3,3I_3,2I_3) \mid I_3 \geq 0\}$
	$(2,4,1)$	$\{(2I_3,4I_3,I_3) \mid I_3 \geq 0\}$	$(2,5,0)$	$\{1/5(2I_2,5I_2,0) \mid I_2 \geq 0\}$
	$(3,4,0)$	$\{1/4(3I_2,4I_2,0) \mid I_2 \geq 0\}$		
8	$(0,1,7)$	$\{1/7(0,I_3,7I_3) \mid I_3 \geq 0\}$	$(0,3,5)$	$\{1/5(0,3I_3,5I_3) \mid I_3 \geq 0\}$
	$(0,5,3)$	$\{1/3(0,5I_3,3I_3) \mid I_3 \geq 0\}$	$(0,7,1)$	$\{(0,7I_3,I_3) \mid I_3 \geq 0\}$
	$(1,2,5)$	$\{1/5(I_3,2I_3,5I_3) \mid I_3 \geq 0\}$	$(1,3,4)$	$\{1/4(I_3,3I_3,4I_3) \mid I_3 \geq 0\}$
	$(1,4,3)$	$\{1/3(I_3,4I_3,3I_3) \mid I_3 \geq 0\}$	$(1,5,2)$	$\{1/2(I_3,5I_3,2I_3) \mid I_3 \geq 0\}$
	$(1,6,1)$	$\{(I_3,6I_3,I_3) \mid I_3 \geq 0\}$	$(1,7,0)$	$\{1/7(I_2,7I_2,0) \mid I_2 \geq 0\}$
	$(2,3,3)$	$\{1/3(2I_3,3I_3,3I_3) \mid I_3 \geq 0\}$	$(2,5,1)$	$\{(2I_3,5I_3,I_3) \mid I_3 \geq 0\}$
	$(3,4,1)$	$\{(3I_3,4I_3,I_3) \mid I_3 \geq 0\}$	$(3,5,0)$	$\{1/5(3I_2,5I_2,0) \mid I_2 \geq 0\}$

Table 5.4: Single resonances with dim $S = 1$.

$\lvert k\rvert$	Resonances	Point (C, A)	Set S
3	$(0,2,1),(1,0,-2)$	$(1.14105, 1.078)$	$\{1/4(I_2-2I_3,4I_2,4I_3) \mid I_2 \geq 2I_3, I_2>0\}$
	$(0,1,2),(1,2,0)$	$(1.15362, 1.14287)$	$\{1/4(2I_2-I_3,4I_2,4I_3) \mid I_3 \leq 2I_2, I_2>0\}$
	$(0,1,2),(1,1,-1)$	$(1.17389, 1.15809)$	$\{1/3(2I_2-I_3,3I_2,3I_3) \mid I_3 \leq 2I_2, I_2>0\}$
	$(1,0,-2),(1,1,-1)$	$(1.18021, 1.13525)$	$\{1/2(I_2-I_3,2I_2,2I_3) \mid I_3 \leq I_2, I_2>0\}$
	$(0,2,1),(1,1,-1)$	$(1.19633, 1.09188)$	$\{1/3(I_2-2I_3,3I_2,3I_3) \mid I_2 \geq 2I_3, I_2>0\}$
	$(0,2,1),(1,2,0)$	$(1.31629, 1.11029)$	$\{1/2(I_2-2I_3,2I_2,2I_3) \mid I_2 \geq 2I_3, I_2>0\}$
	$(1,-1,-1),(1,2,0)$	$(1.60773, 1.08671)$	$\{1/2(I_2-3I_3,2I_2,2I_3) \mid I_2 \geq 3I_3, I_2>0\}$
	$(1,2,0),(2,0,-1)$	$(2.06397, 1.07194)$	$\{1/2(I_2-4I_3,2I_2,2I_3) \mid I_2 \geq 4I_3, I_2>0\}$
4	$(0,3,1),(1,0,-3)$	$(1.07076, 1.03798)$	$\{1/9(I_2-3I_3,9I_2,9I_3) \mid I_2 \geq 3I_3, I_2>0\}$
	$(0,3,1),(1,1,-2)$	$(1.09762, 1.04473)$	$\{1/7(I_2-3I_3,7I_2,7I_3) \mid I_2 \geq 3I_3, I_2>0\}$
	$(0,1,3),(1,3,0)$	$(1.1186, 1.11601)$	$\{1/9(3I_2-I_3,9I_2,9I_3) \mid I_3 \leq 3I_2, I_2>0\}$
	$(1,0,-3),(1,1,-2)$	$(1.13393, 1.1099)$	$\{1/3(I_2-I_3,3I_2,3I_3) \mid I_3 \leq I_2, I_2>0\}$
	$(0,1,3),(1,2,-1)$	$(1.13783, 1.13403)$	$\{1/7(3I_2-I_3,7I_2,7I_3) \mid I_3 \leq 3I_2, I_2>0\}$
	$(0,3,1),(1,2,-1)$	$(1.15439, 1.05456)$	$\{1/5(I_2-3I_3,5I_2,5I_3) \mid I_2 \geq 3I_3, I_2>0\}$
	$(0,1,3),(1,2,1)$	$(1.16099, 1.15492)$	$\{1/5(3I_2-I_3,5I_2,5I_3) \mid I_3 \leq 3I_2, I_2>0\}$
	$(1,0,-3),(1,2,1)$	$(1.17389, 1.15809)$	$\{1/3(2I_2-I_3,3I_2,3I_3) \mid I_3 \leq 2I_3, I_2>0\}$
	$(1,-1,-2),(1,3,0)$	$(1.19633, 1.09188)$	$\{1/3(I_2-2I_3,3I_2,3I_3) \mid I_2 \geq 2I_3, I_2>0\}$
	$(0,3,1),(1,3,0)$	$(1.32794, 1.07139)$	$\{1/3(I_2-3I_3,3I_2,3I_3) \mid I_2 \geq 3I_3, I_2>0\}$
	$(0,3,1),(2,1,-1)$	$(1.60773, 1.08671)$	$\{1/2(I_2-3I_3,2I_2,2I_3) \mid I_2 \geq 3I_3, I_2>0\}$
	$(1,-2,-1),(1,3,0)$	$(1.83326, 1.0462)$	$\{1/3(I_2-5I_3,3I_2,3I_3) \mid I_2 \geq 5I_3, I_2>0\}$

Table 5.5: Multiple resonances of the same order with dim $S = 2$.

$\|k\|$	Resonances	Point (C, A)	Set S
5	$(0,4,1),(1,0,-4)$	$(1.04137, 1.02192)$	$\{1/16(I_2-4I_3, 16I_2, 16I_3) \mid I_2 \geq 4I_3, I_2 > 0\}$
	$(0,4,1),(1,1,-3)$	$(1.05431, 1.02515)$	$\{1/13(I_2-4I_3, 13I_2, 13I_3) \mid I_2 \geq 4I_3, I_2 > 0\}$
	$(0,4,1),(1,-1,-3)$	$(1.06825, 1.02788)$	$\{1/11(I_2-4I_3, 11I_2, 11I_3) \mid I_2 \geq 4I_3, I_2 > 0\}$
	$(0,4,1),(1,2,-2)$	$(1.07807, 1.02949)$	$\{1/10(I_2-4I_3, 10I_2, 10I_3) \mid I_2 \geq 4I_3, I_2 > 0\}$
	$(0,3,2),(1,0,-4)$	$(1.08076, 1.064)$	$\{1/24(4I_2-6I_3, 24I_2, 24I_3) \mid 2I_2 \geq 3I_3, I_2 > 0\}$
	$(1,-1,-3),(1,2,-2)$	$(1.08211, 1.04107)$	$\{1/8(I_2-3I_3, 8I_2, 8I_3) \mid I_2 \geq 3I_3, I_2 > 0\}$
	$(0,3,2),(1,1,-3)$	$(1.08791, 1.06831)$	$\{1/11(2I_2-3I_3, 11I_2, 11I_3) \mid 2I_2 \geq 3I_3, I_2 > 0\}$
	$(0,3,2),(1,2,-2)$	$(1.09646, 1.07322)$	$\{1/10(2I_2-3I_3, 10I_2, 10I_3) \mid 2I_2 \geq 3I_3, I_2 > 0\}$
	$(0,1,4),(1,4,0)$	$(1.09664, 1.09573)$	$\{1/16(4I_2-I_3, 16I_2, 16I_3) \mid I_3 \leq 4I_2, I_2 > 0\}$
	$(0,2,3),(1,4,0)$	$(1.10039, 1.09384)$	$\{1/24(6I_2-4I_3, 24I_2, 24I_3) \mid 3I_2 \geq 2I_3, I_2 > 0\}$
	$(1,0,-4),(1,1,-3)$	$(1.10607, 1.09115)$	$\{1/4(I_2-I_3, 4I_2, 4I_3) \mid I_3 \leq I_2, I_2 > 0\}$
	$(0,3,2),(1,3,-1)$	$(1.10685, 1.07885)$	$\{1/9(2I_2-3I_3, 9I_2, 9I_3) \mid 2I_2 \geq 3I_3, I_2 > 0\}$
	$(0,2,3),(1,3,-1)$	$(1.1073, 1.09973)$	$\{1/11(3I_2-2I_3, 11I_2, 11I_3) \mid 3I_2 \geq 2I_3, I_2 > 0\}$
	$(1,-1,-3),(1,3,-1)$	$(1.10985, 1.06762)$	$\{1/5(I_2-2I_3, 5I_2, 5I_3) \mid I_2 \geq 2I_3, I_2 > 0\}$
	$(0,1,4),(1,3,-1)$	$(1.1118, 1.11052)$	$\{1/13(4I_2-I_3, 13I_2, 13I_3) \mid I_3 \leq 4I_2, I_2 > 0\}$
	$(0,2,3),(1,2,-2)$	$(1.11517, 1.10631)$	$\{1/10(3I_2-2I_3, 10I_2, 10I_3) \mid 3I_2 \geq 2I_3, I_2 > 0\}$
	$(0,3,2),(1,4,0)$	$(1.11977, 1.08538)$	$\{1/8(2I_2-3I_3, 8I_2, 8I_3) \mid 2I_2 \geq 3I_3, I_2 > 0\}$
	$(0,2,3),(1,1,-3)$	$(1.12417, 1.11368)$	$\{1/9(3I_2-2I_3, 9I_2, 9I_3) \mid 3I_2 \geq 2I_3, I_2 > 0\}$
	$(0,1,4),(1,3,1)$	$(1.12437, 1.1227)$	$\{1/11(4I_2-I_3, 11I_2, 11I_3) \mid I_3 \leq 4I_2, I_2 > 0\}$
	$(1,-1,-3),(1,4,0)$	$(1.12594, 1.08306)$	$\{1/12(3I_2-5I_3, 12I_2, 12I_3) \mid 3I_2 \geq 5I_3, I_2 > 0\}$
	$(1,2,-2),(1,3,1)$	$(1.12766, 1.12455)$	$\{1/8(3I_2-I_3, 8I_2, 8I_3) \mid I_3 \leq 3I_2, I_2 > 0\}$
	$(0,1,4),(1,2,-2)$	$(1.13152, 1.12959)$	$\{1/10(4I_2-I_3, 10I_2, 10I_3) \mid I_3 \leq 4I_2, I_2 > 0\}$
	$(0,4,1),(1,3,-1)$	$(1.13278, 1.0358)$	$\{1/7(I_2-4I_3, 7I_2, 7I_3) \mid I_2 \geq 4I_3, I_2 > 0\}$
	$(0,2,3),(1,0,-4)$	$(1.13452, 1.12191)$	$\{1/8(3I_2-2I_3, 8I_2, 8I_3) \mid 3I_2 \geq 2I_3, I_2 > 0\}$
	$(1,1,-3),(1,3,1)$	$(1.13576, 1.12799)$	$\{1/5(2I_2-I_3, 5I_2, 5I_3) \mid I_3 \leq 2I_2, I_2 > 0\}$
	$(0,3,2),(1,-1,-3)$	$(1.13626, 1.09298)$	$\{1/7(2I_2-3I_3, 7I_2, 7I_3) \mid 2I_2 \geq 3I_3, I_2 > 0\}$
	$(1,0,-4),(1,3,1)$	$(1.14181, 1.12987)$	$\{1/12(5I_2-3I_3, 12I_2, 12I_3) \mid 5I_2 \geq 3I_3, I_2 > 0\}$
	$(1,-2,-2),(1,3,-1)$	$(1.14628, 1.02656)$	$\{1/8(I_2-5I_3, 8I_2, 8I_3) \mid I_2 \geq 5I_3, I_2 > 0\}$
	$(0,2,3),(1,3,1)$	$(1.14646, 1.13101)$	$\{1/7(3I_2-2I_3, 7I_2, 7I_3) \mid 3I_2 \geq 2I_3, I_2 > 0\}$
	$(0,1,4),(1,1,-3)$	$(1.15606, 1.15288)$	$\{1/7(4I_2-I_3, 7I_2, 7I_3) \mid I_3 \leq 4I_2, I_2 > 0\}$

Table 5.6: Multiple resonances of the same order 5 with dim $S = 2$.

$\lvert k\rvert$	Resonances	Point (C, A)	Set S
5	$(1,3,-1),(2,0,-3)$	$(1.15929, 1.0202)$	$\{1/9(I_2-6I_3, 9I_2, 9I_3) \mid I_2 \geq 6I_3, I_2>0\}$
	$(1,1,-3),(1,2,2)$	$(1.16017, 1.15789)$	$\{1/8(5I_2-I_3, 8I_2, 8I_3) \mid I_3 \leq 5I_2, I_2>0\}$
	$(1,1,-3),(2,3,0)$	$(1.16278, 1.16106)$	$\{1/9(6I_2-I_3, 9I_2, 9I_3) \mid I_3 \leq 6I_2, I_2>0\}$
	$(0,1,4),(1,2,2)$	$(1.16448, 1.1606)$	$\{1/6(4I_2-I_3, 6I_2, 6I_3) \mid I_3 \leq 4I_2, I_2>0\}$
	$(1,0,-4),(2,3,0)$	$(1.16835, 1.15955)$	$\{1/12(8I_2-3I_3, 12I_2, 12I_3) \mid 8I_2 \geq 3I_3, I_2>0\}$
	$(0,4,1),(1,-2,-2)$	$(1.16932, 1.03872)$	$\{1/6(I_2-4I_3, 6I_2, 6I_3) \mid I_2 \geq 4I_3, I_2>0\}$
	$(1,0,-4),(1,2,2)$	$(1.17235, 1.16434)$	$\{1/4(3I_2-I_3, 4I_2, 4I_3) \mid I_3 \leq 3I_2, I_2>0\}$
	$(0,1,4),(2,2,-1)$	$(1.17434, 1.16891)$	$\{1/9(8I_2-2I_3, 9I_2, 9I_3) \mid I_3 \leq 4I_2, I_2>0\}$
	$(1,0,-4),(2,2,-1)$	$(1.17535, 1.16837)$	$\{1/8(7I_2-2I_3, 8I_2, 8I_3) \mid 7I_2 \geq 2I_3, I_2>0\}$
	$(1,2,2),(2,2,-1)$	$(1.17934, 1.16626)$	$\{1/6(5I_2-2I_3, 6I_2, 6I_3) \mid 5I_2 \geq 2I_3, I_2>0\}$
	$(1,-1,-3),(1,3,1)$	$(1.18021, 1.13525)$	$\{1/2(I_2-I_3, 2I_2, 2I_3) \mid I_3 \leq I_2, I_2>0\}$
	$(0,2,3),(2,3,0)$	$(1.18411, 1.15553)$	$\{1/9(2\,(3I_2-2I_3)\,, 9I_2, 9I_3) \mid 3I_2 \geq 2I_3, I_2>0\}$
	$(1,4,0),(2,0,-3)$	$(1.18413, 1.06701)$	$\{1/12(3I_2-8I_3, 12I_2, 12I_3) \mid 3I_2 \geq 8I_3, I_2>0\}$
	$(0,2,3),(2,2,-1)$	$(1.19261, 1.15949)$	$\{1/4(3I_2-2I_3, 4I_2, 4I_3) \mid 3I_2 \geq 2I_3, I_2>0\}$
	$(1,-1,-3),(2,3,0)$	$(1.19793, 1.15232)$	$\{1/9(6I_2-5I_3, 9I_2, 9I_3) \mid 6I_2 \geq 5I_3, I_2>0\}$
	$(0,2,3),(2,1,-2)$	$(1.20092, 1.16212)$	$\{1/7(2\,(3I_2-2I_3)\,, 7I_2, 7I_3) \mid 3I_2 \geq 2I_3, I_2>0\}$
	$(1,-1,-3),(2,2,-1)$	$(1.20109, 1.15536)$	$\{1/7(5I_2-4I_3, 7I_2, 7I_3) \mid 5I_2 \geq 4I_3, I_2>0\}$
	$(1,-1,-3),(2,1,-2)$	$(1.20513, 1.15925)$	$\{1/5(4I_2-3I_3, 5I_2, 5I_3) \mid 4I_2 \geq 3I_3, I_2>0\}$
	$(0,3,2),(2,0,-3)$	$(1.20757, 1.11783)$	$\{1/9(2\,(2I_2-3I_3)\,, 9I_2, 9I_3) \mid 2I_2 \geq 3I_3, I_2>0\}$
	$(1,-2,-2),(1,4,0)$	$(1.21297, 1.06159)$	$\{1/4(I_2-3I_3, 4I_2, 4I_3) \mid I_2 \geq 3I_3, I_2>0\}$
	$(1,3,1),(2,0,-3)$	$(1.21369, 1.13621)$	$\{1/9(5I_2-6I_3, 9I_2, 9I_3) \mid 5I_2 \geq 6I_3, I_2>0\}$
	$(2,0,-3),(2,1,-2)$	$(1.21584, 1.1485)$	$\{1/3(2\,(I_2-I_3)\,, 3I_2, 3I_3) \mid I_3 \leq I_2, I_2>0\}$
	$(1,3,1),(2,1,-2)$	$(1.22445, 1.13621)$	$\{1/7(4I_2-5I_3, 7I_2, 7I_3) \mid 4I_2 \geq 5I_3, I_2>0\}$
	$(0,3,2),(2,1,-2)$	$(1.23161, 1.1237)$	$\{1/4(2I_2-3I_3, 4I_2, 4I_3) \mid 2I_2 \geq 3I_3, I_2>0\}$
	$(1,3,1),(2,2,-1)$	$(1.24515, 1.13598)$	$\{1/5(3I_2-4I_3, 5I_2, 5I_3) \mid 3I_2 \geq 4I_3, I_2>0\}$
	$(1,-2,-2),(2,1,-2)$	$(1.25221, 1.0813)$	$\{1/6(2I_2-5I_3, 6I_2, 6I_3) \mid 2I_2 \geq 5I_3, I_2>0\}$
	$(0,3,2),(2,2,-1)$	$(1.26165, 1.12948)$	$\{1/7(2\,(2I_2-3I_3)\,, 7I_2, 7I_3) \mid 2I_2 \geq 3I_3, I_2>0\}$
	$(1,4,0),(2,1,-2)$	$(1.26609, 1.05408)$	$\{1/8(2I_2-7I_3, 8I_2, 8I_3) \mid 2I_2 \geq 7I_3, I_2>0\}$
	$(0,4,1),(2,1,-2)$	$(1.27154, 1.04478)$	$\{1/9(2\,(I_2-4I_3)\,, 9I_2, 9I_3) \mid I_2 \geq 4I_3, I_2>0\}$
	$(0,3,2),(1,3,1)$	$(1.29998, 1.13455)$	$\{1/3(2I_2-3I_3, 3I_2, 3I_3) \mid 2I_2 \geq 3I_3, I_2>0\}$

Table 5.7: Multiple resonances of the same order 5 with dim $S = 2$.

$\lvert k\rvert$	Resonances	Point (C, A)	Set S
	$(1,-2,-2),(2,2,-1)$	$(1.31629, 1.11029)$	$\{1/2(I_2-2I_3, 2I_2, 2I_3) \mid I_2 \geq 2I_3, I_2>0\}$
	$(0,4,1),(1,4,0)$	$(1.33123, 1.04762)$	$\{1/4(I_2-4I_3, 4I_2, 4I_3) \mid I_2 \geq 4I_3, I_2>0\}$
	$(1,-2,-2),(2,3,0)$	$(1.3627, 1.12698)$	$\{1/6(4I_2-7I_3, 6I_2, 6I_3) \mid 4I_2 \geq 7I_3 \mid I_2>0\}$
	$(1,-2,-2),(1,3,1)$	$(1.38029, 1.13164)$	$\{1/4(3I_2-5I_3, 4I_2, 4I_3) \mid 3I_2 \geq 5I_3, I_2>0\}$
	$(1,4,0),(2,-1,-2)$	$(1.40824, 1.04227)$	$\{1/8(2I_2-9I_3, 8I_2, 8I_3) \mid 2I_2 \geq 9I_3, I_2>0\}$
	$(0,4,1),(2-1,-2)$	$(1.41581, 1.05123)$	$\{1/7(2\,(I_2-4I_3)\,, 7I_2, 7I_3) \mid I_2 \geq 4I_3, I_2>0\}$
	$(2,-1,-2),(2,2,-1)$	$(1.43332, 1.07811)$	$\{1/5(2\,(I_2-3I_3)\,, 5I_2, 5I_3) \mid I_2 \geq 3I_3, I_2>0\}$
	$(1,3,1),(2,-1,-2)$	$(1.43375, 1.12953)$	$\{1/5(4I_2-7I_3, 5I_2, 5I_3) \mid 4I_2 \geq 7I_3, I_2>0\}$
5	$(2,-1,-2),(2,3,0)$	$(1.44114, 1.11963)$	$\{1/3(2\,(I_2-2I_3)\,, 3I_2, 3I_3) \mid I_2 \geq 2I_3, I_2>0\}$
	$(1,3,1),(3,0,-2)$	$(1.47118, 1.12801)$	$\{1/6(5I_2-9I_3, 6I_2, 6I_3) \mid 5I_2 \geq 9I_3, I_2>0\}$
	$(2,3,0),(3,0,-2)$	$(1.53601, 1.11281)$	$\{1/6(4I_2-9I_3, 6I_2, 6I_3) \mid 4I_2 \geq 9I_3, I_2>0\}$
	$(0,4,1),(2,2,-1)$	$(1.54112, 1.05599)$	$\{1/3(I_2-4I_3, 3I_2, 3I_3) \mid I_2 \geq 4I_3, I_2>0\}$
	$(0,4,1),(3,0,-2)$	$(1.6611, 1.06013)$	$\{1/8(3\,(I_2-4I_3)\,, 8I_2, 8I_3) \mid I_2 \geq 4I_2I_3>0\}$
	$(1,4,0),(2,2,-1)$	$(1.70724, 1.03158)$	$\{1/4(I_2-6I_3, 4I_2, 4I_3) \mid I_2 \geq 6I_3, I_2>0\}$
	$(1,-3,-1),(3,0,-2)$	$(1.73031, 1.01519)$	$\{1/6(I_2-9I_3, 6I_2, 6I_3) \mid I_2 \geq 9I_3, I_2>0\}$
	$(1,-3,-1),(2,2,-1)$	$(1.82073, 1.01993)$	$\{1/5(I_2-8I_3, 5I_2, 5I_3) \mid I_2 \geq 8I_3, I_2>0\}$
	$(1,-3,-1),(1,4,0)$	$(1.96045, 1.02738)$	$\{1/4(I_2-7I_3, 4I_2, 4I_3) \mid I_2 \geq 7I_3, I_2>0\}$

Table 5.8: Multiple resonances of the same order 5 with dim $S=2$.

$\|k\|$		Vectors	Point	Set S
3	4	$(1,2,0),(0,1,3)$	$(1.14906, 1.14432)$	$\{1/6(3I_2-I_3, 6I_2, 6I_3) \mid I_3 \leq 3I_2, I_2>0\}$
		$(1,1,-1),(0,1,3)$	$(1.17235, 1.16434)$	$\{1/4(3I_2-I_3, 4I_2, 4I_3) \mid I_3 \leq 3I_2, I_2>0\}$
	5	$(1,2,0),(0,1,4)$	$(1.14749, 1.14484)$	$\{1/8(4I_2-I_3, 8I_2, 8I_3) \mid I_3 \leq 4I_2, I_2>0\}$
		$(1,1,-1),(0,1,4)$	$(1.17173, 1.16691)$	$\{1/5(4I_2-I_3, 5I_2, 5I_3) \mid I_3 \leq 4I_2, I_2>0\}$
	6	$(1,2,0),(0,1,5)$	$(1.14677, 1.14507)$	$\{1/10(5I_2-I_3, 10I_2, 10I_3) \mid I_3 \leq 5I_2, I_2>0\}$
		$(1,1,-1),(0,1,5)$	$(1.17143, 1.1682)$	$\{1/6(5I_2-I_3, 6I_2, 6I_3) \mid I_3 \leq 5I_2, I_2>0\}$
	7	$(1,2,0),(0,1,6)$	$(1.14638, 1.1452)$	$\{1/12(6I_2-I_3, 12I_2, 12I_3) \mid I_3 \leq 6I_2, I_2>0\}$
		$(1,1,-1),(0,1,6)$	$(1.17125, 1.16895)$	$\{1/7(6I_2-I_3, 7I_2, 7I_3) \mid I_3 \leq 6I_2, I_2>0\}$
	8	$(1,2,0),(0,1,7)$	$(1.14615, 1.14528)$	$\{1/14(7I_2-I_3, 14I_2, 14I_3) \mid I_3 \leq 7I_2, I_2>0\}$
		$(1,1,-1),(0,1,7)$	$(1.17115, 1.16941)$	$\{1/8(7I_2-I_3, 8I_2, 8I_3) \mid I_3 \leq 7I_2, I_2>0\}$
4	5	$(1,3,0),(0,1,4)$	$(1.11781, 1.11635)$	$\{1/12(4I_2-I_3, 12I_2, 12I_3) \mid I_3 \leq 4I_2, I_2>0\}$
		$(1,2,-1),(0,1,4)$	$(1.13924, 1.137)$	$\{1/9(4I_2-I_3, 9I_2, 9I_3) \mid I_3 \leq 4I_2, I_2>0\}$
		$(1,2,1),(0,1,4)$	$(1.15606, 1.15288)$	$\{1/7(4I_2-I_3, 7I_2, 7I_3) \mid I_3 \leq 4I_2, I_2>0\}$
		$(1,1,-2),(0,1,4)$	$(1.16448, 1.1606)$	$\{1/6(4I_2-I_3, 6I_2, 6I_3) \mid I_3 \leq 4I_2, I_2>0\}$
	6	$(1,3,0),(0,1,5)$	$(1.11744, 1.11651)$	$\{1/15(5I_2-I_3, 15I_2, 15I_3) \mid I_3 \leq 5I_2, I_2>0\}$
		$(1,2,-1),(0,1,5)$	$(1.14024, 1.13875)$	$\{1/11(5I_2-I_3, 11I_2, 11I_3) \mid I_3 \leq 5I_2, I_2>0\}$
		$(1,2,1),(0,1,5)$	$(1.15349, 1.15153)$	$\{1/9(5I_2-I_3, 9I_2, 9I_3) \mid I_3 \leq 5I_2, I_2>0\}$
		$(1,1,-2),(0,1,5)$	$(1.1664, 1.1637)$	$\{1/7(5I_2-I_3, 7I_2, 7I_3) \mid I_3 \leq 5I_2, I_2>0\}$
	7	$(1,3,0),(0,1,6)$	$(1.11724, 1.1166)$	$\{1/18(6I_2-I_3, 18I_2, 18I_3) \mid I_3 \leq 6I_2, I_2>0\}$
		$(1,2,-1),(0,1,6)$	$(1.14096, 1.13991)$	$\{1/13(6I_2-I_3, 13I_2, 13I_3) \mid I_3 \leq 6I_2, I_2>0\}$
		$(1,2,1),(0,1,6)$	$(1.15192, 1.15059)$	$\{1/11(6I_2-I_3, 11I_2, 11I_3) \mid I_3 \leq 6I_2, I_2>0\}$
		$(1,1,-2),(0,1,6)$	$(1.16756, 1.16558)$	$\{1/8(6I_2-I_3, 8I_2, 8I_3) \mid I_3 \leq 6I_2, I_2>0\}$
	8	$(1,3,0),(0,1,7)$	$(1.11712, 1.11665)$	$\{1/21(7I_2-I_3, 21I_2, 21I_3) \mid I_3 \leq 7I_2, I_2>0\}$
		$(1,2,-1),(0,1,7)$	$(1.14152, 1.14073)$	$\{1/15(7I_2-I_3, 15I_2, 15I_3) \mid I_3 \leq 7I_2, I_2>0\}$
		$(1,2,1),(0,1,7)$	$(1.15086, 1.1499)$	$\{1/13(7I_2-I_3, 13I_2, 13I_3) \mid I_3 \leq 7I_2, I_2>0\}$
		$(1,1,-2),(0,1,7)$	$(1.16832, 1.1668)$	$\{1/9(7I_2-I_3, 9I_2, 9I_3) \mid I_3 \leq 7I_2, I_2>0\}$

Table 5.9: Multiple resonances of different order with dim $S = 2$.

5.2 Analysis of the Lie stability

5.2.1 Dimension of set S is zero

According Remark 2.4.2 in the case of single resonance and the components of its resonance vector change of sign, then the set S is null and we have Lie stability. This situation for the equilibrium point P_3 for resonance up to order 8 correspond to the 124 cases that appear in the Table 5.2 and 5.3.

5.2.2 Dimension of set S is one

5.2.3 Dimension of set S is two

We analyze some cases of multiple resonances of the same order 3, namely,

1) $(0,1,2)$ and $(1,1,-1)$, 2) $(1,0,-2)$ and $(1,1,-1)$,

3) $(1,-1,-1)$ and $(1,2,0)$, 4) $(1,2,0)$ and $(2,0,-1)$.

For the previous cases, the Hamiltonian in its Lie normal form (for more details see [55]) up to order three is

$$\mathcal{H} = \mathcal{H}_2 + \mathcal{H}_3, \tag{5.1}$$

whose quadratic and cubic part is given in Table 5.10.

Case	$\mathcal{H}_2$	$\mathcal{H}_3$
1	$0.65315 I_1 - 0.435433 I_2 + 0.217717 I_3$	$1.5061\sqrt{I_1 I_2 I_3}\sin(\varphi_1+\varphi_2-\varphi_3)$
2	$0.734514 I_1 - 0.367257 I_2 + 0.367257 I_3$	$1.10453\sqrt{I_1 I_2 I_3}\sin(\varphi_1+\varphi_2-\varphi_3)$
3	$\frac{1}{3}(2.50044 I_1 - 1.25022 I_2 + 3.75066 I_3)$	$-1.97211\sqrt{I_1 I_2 I_3}\sin(\varphi_1-\varphi_2-\varphi_3)$
4	$\frac{1}{4}(3.45027 I_1 - 1.72513 I_2 + 6.90053 I_3)$	$1.43573 I_1\sqrt{I_3}\sin(2\varphi_1-\varphi_3)$

Table 5.10: Lie normal form of the terms of order three.

After normalizing, we obtain Hamiltonian with only resonance vector whose components change sign, therefore, the origin of $\mathbb{R}^6$ is Lie stable for the Hamiltonian system associated to 5.1.

We analyze some cases of multiple resonances of the same order 5, specifically

1)	$(1,-1,-3)$ and $(1,2,-2)$,	2)	$(0,3,2)$ and $(1,3,-1)$,
3)	$(0,1,4)$ and $(1,3,-1)$,	4)	$(0,3,2)$ and $(1,-1,-3)$,
5)	$(1,-2,-2)$ and $(1,3,-1)$,	6)	$(1,1,-3)$ and $(1,2,2)$,
7)	$(1,0,-4)$ and $(2,2,-1)$,	8)	$(1,2,2)$ and $(2,2,-1)$,
9)	$(1,4,0)$ and $(2,0,-3)$,	10)	$(1,-1,-3)$ and $(2,3,0)$,
11)	$(1,-1,-3)$ and $(2,1,-2)$,	12)	$(0,3,2)$ and $(2,0,-3)$,
13)	$(2,-1,-2)$ and $(2,2,-1)$,	14)	$(1,-3,-1)$ and $(3,0,-2)$.

For the previous cases, the Hamiltonian in its Lie normal form up to order five is

$$\mathcal{H} = \mathcal{H}_2 + \mathcal{H}_5, \tag{5.2}$$

whose quadratic and cubic part is given in Table 5.11.

Case	$\mathcal{H}_2$	$\mathcal{H}_5$
1	$\frac{1}{3}(2.80695I_1 - 0.350869I_2 + 1.05261I_3)$	$0.626729\sin(\varphi_1 - \varphi_2 - 3\varphi_3)\sqrt{I_1 I_2} I_3^{3/2}$
2	$\frac{1}{3}(2.60845I_1 - 0.579655I_2 + 0.869482I_3)$	$5.07783\sin(\varphi_1 + 3\varphi_2 - \varphi_3)\sqrt{I_1 I_3} I_2^{3/2}$
3	$0.806372I_1 - 0.248115I_2 + 0.0620286I_3$	$11.2652\sin(\varphi_1 + 3\varphi_2 - \varphi_3)\sqrt{I_1 I_3} I_2^{3/2}$
4	$\frac{1}{3}(2.52251I_1 - 0.720718I_2 + 1.08108I_3)$	$-5.79936\sin(\varphi_1 - \varphi_2 - 3\varphi_3)\sqrt{I_1 I_2} I_3^{3/2}$
5	$\frac{1}{5}(4.79431I_1 - 0.599289I_2 + 2.99644I_3)$	$0.591001\sin(\varphi_1 + 3\varphi_2 - \varphi_3)\sqrt{I_1 I_3} I_2^{3/2}$
6	$0.662613I_1 - 0.414133I_2 + 0.0828266I_3$	$-45.8689\sin(\varphi_1 + \varphi_2 - 3\varphi_3)\sqrt{I_1 I_2} I_3^{3/2}$
7	$\frac{1}{7}(4.05244I_1 - 3.54588I_2 + 1.01311I_3)$	$-1631.71\sin(2\varphi_1 + 2\varphi_2 - \varphi_3) I_1 I_2 \sqrt{I_3}$
8	$\frac{1}{2}(1.18889I_1 - 0.990743I_2 + 0.396297I_3)$	$-548.516\sin(2\varphi_1 + 2\varphi_2 - \varphi_3) I_1 I_2 \sqrt{I_3}$
9	$\frac{1}{8}(7.11317I_1 - 1.77829I_2 + 4.74212I_3)$	$-8.82407\sin(2\varphi_1 - 3\varphi_3) I_1 I_3^{3/2}$
10	$\frac{1}{5}(3.32922I_1 - 2.21948I_2 + 1.84957I_3)$	$28.9564\sin(\varphi_1 - \varphi_2 - 3\varphi_3)\sqrt{I_1 I_2} I_3^{3/2}$
11	$\frac{1}{3}(1.85488I_1 - 1.4839I_2 + 1.11293I_3)$	$37.3351\sin(\varphi_1 - \varphi_2 - 3\varphi_3)\sqrt{I_1 I_2} I_3^{3/2}$
12	$\frac{1}{4}(3.11316I_1 - 1.38363I_2 + 2.07544I_3)$	$-24.4457\sin(2\varphi_1 - 3\varphi_3) I_1 I_3^{3/2}$
13	$\frac{1}{6}(5.16142I_1 - 2.06457I_2 + 6.1937I_3)$	$4.40635\sin(2\varphi_1 + 2\varphi_2 - \varphi_3) I_1 I_2 \sqrt{I_3}$
14	$0.976472I_1 - 0.162745I_2 + 1.46471I_3$	$-0.133848\sin(\varphi_1 - 3\varphi_2 - \varphi_3)\sqrt{I_1 I_3} I_2^{3/2}$

Table 5.11: Lie normal form of the terms of order five.

After normalizing, we obtain Hamiltonian with only resonance vector whose components change sign, therefore, the origin of $\mathbb{R}^6$ is Lie stable for the Hamiltonian system associated to (5.2).

5.3 Instability at the point P_1 in the case of single resonance

We compute that in the region II there are exactly seven resonance curves of order 3, which are characterized by $2\omega_1 - \omega_3 = 0$, $\omega_1 - 2\omega_2 = 0$, $\omega_1 - \omega_2 - \omega_3 = 0$, $\omega_1 - 2\omega_3 = 0$, $\omega_1 + \omega_2 - \omega_3 = 0$, $2\omega_2 - \omega_3 = 0$ and $\omega_2 - 2\omega_3 = 0$. As we see in Theorem 3.2 a necessary condition to have instability in the case of simple resonance is that all the components of the resonance vector must have the same sign. So we need to analyze the following situations $c13 : \omega_1 - 2\omega_2 = 0$, $c23 : 2\omega_2 - \omega_3 = 0$ and $c33 : \omega_2 - 2\omega_3 = 0$. (See Figure 5.4). Here the associated vectors of resonances are $(1, 2, 0)$, $(0, 2, 1)$ and $(0, 1, 2)$, respectively.

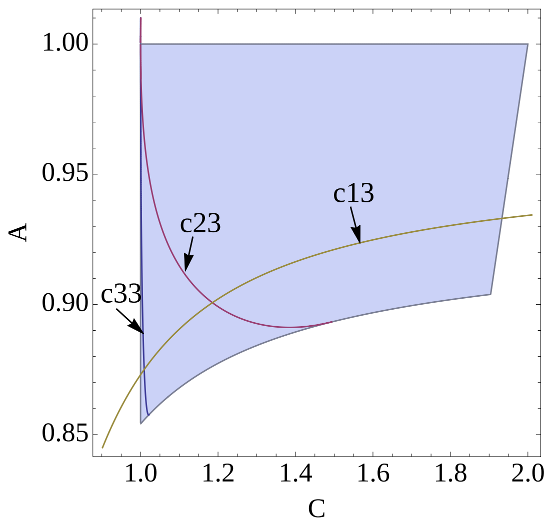


Figure 5.4: Resonance curves of order three in region II.

First, we analyze the case $k = (0, 2, 1)$. With the help of Mathematica software, we compute that the Lie normalized Hamiltonian up to order three is

$$\mathcal{H} = \omega_1 I_1 - \frac{1}{2}\omega_3 I_2 + \omega_3 I_3 + \gamma_0 I_2 I_3^{1/2} \sin(2\varphi_2 + \varphi_3) + \cdots , \tag{5.3}$$

where

$$\gamma_0 = \frac{\omega_2 \left(\omega_1^2 \left(\omega_2^2 + 3\right) - 4\right) \omega_3 - 4 \left(\omega_1^2 - 1\right) \left(\omega_2^2 - 1\right)}{4\sqrt{2} \left(\omega_1^2 - \omega_2^2\right) \sqrt{\omega_3}}.$$

On the resonance curve $2\omega_2 - \omega_3 = 0$, we can write $\gamma_0 = \gamma_0(C)$, i.e., γ_0 as function of the parameter C. In Figure 5.5 we draw γ_0, so $\gamma_0 \neq 0$. Therefore, by Theorem 3.2 the equilibrium point P_1 is unstable in the Liapunov sense.

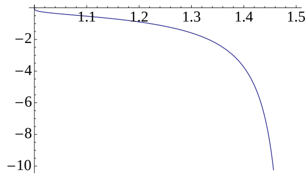


Figure 5.5: Graph of the curve $\gamma_0(C)$.

In the other two cases $k = (1, 2, 0)$ and $k = (0, 1, 2)$, we have that the cubic part of the Lie normalized Hamiltonian is null, so we can not apply Theorem 3.2. Thus, the stability problem must be studied using terms of upper order.

Now, we will study the cases of single resonance of order four in region II. We have that there are exactly twelve resonance curves of order 4, which are characterized by $2\omega_1 - \omega_2 - \omega_3 = 0$, $2\omega_1 - 2\omega_3 = 0$, $\omega_1 - 3\omega_2 = 0$, $\omega_1 - 2\omega_2 + \omega_3 = 0$, $\omega_1 - 2\omega_2 - \omega_3 = 0$, $\omega_1 - \omega_2 - 2\omega_3 = 0$, $\omega_1 - 3\omega_3 = 0$, $\omega_1 + \omega_2 - 2\omega_3 = 0$, $\omega_1 + 2\omega_2 - \omega_3 = 0$, $-3\omega_2 + \omega_3 = 0$, $-2\omega_2 + 2\omega_3 = 0$, $-\omega_2 + 3\omega_3 = 0$. Again, as we see in Theorem 3.2, a necessary condition to have instability in the case of simple resonance is that all the components of the resonance vector have the same sign. So we need to analyze the following cases $c14 : \omega_1 - 3\omega_2 = 0$, $c24 : \omega_1 - 2\omega_2 + \omega_3 = 0$, $c34 : -3\omega_2 + \omega_3 = 0$, $c44 : 2\omega_2 - 2\omega_3 = 0$ and $c54 : \omega_2 - 3\omega_3 = 0$. Here the associated vectors of resonances are $(1, 3, 0)$, $(1, 2, 1)$, $(0, 3, 1)$, $(0, 2, 2)$ and $(0, 1, 3)$. See Figure 5.6 for the previous resonance curves.

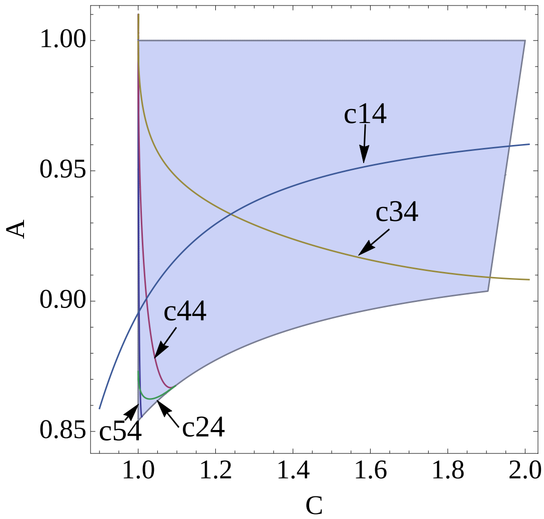


Figure 5.6: Resonance curves of order four in region II associated to the cases $(1, 3, 0)$, $(1, 2, 1)$, $(0, 3, 1)$, $(0, 2, 2)$ and $(0, 1, 3)$.

We start studying the case $k = (1, 3, 0)$. Here the Lie normalized Hamiltonian up to order four is

$$\begin{aligned}\mathcal{H} &= 3\omega_2 I_1 - \omega_2 I_2 + \omega_3 I_3 + a_{11} I_1^2 + a_{12} I_1 I_2 + a_{13} I_1 I_3 + a_{14} I_2^2 + \\ & a_{15} I_2 I_3 + a_{16} I_3^2 + \gamma_1 I_1^{1/2} I_2^{3/2} \cos(\varphi_1 + 3\varphi_2) + \cdots ,\end{aligned} \tag{5.4}$$

where

$$\begin{aligned}
a_{11} &= \tfrac{1}{\delta_{11}} (-8\omega_1^{16} + 3\omega_3^2\omega_1^{14} + 432\omega_1^{14} - 151\omega_3^2\omega_1^{12} - 6180\omega_1^{12} + 1748\omega_3^2\omega_1^{10} + \\
& 42604\omega_1^{10} - 8676\omega_3^2\omega_1^8 + 176704\omega_1^8 - 50940\omega_3^2\omega_1^6 - 2110896\omega_1^6 + 496368\omega_3^2\omega_1^4 + \\
& 2468880\omega_1^4 - 571536\omega_3^2\omega_1^2 - 991440\omega_1^2 + 419904\omega_3^2 + 419904) , \\
a_{12} &= \tfrac{1}{\delta_{12}} (-64\omega_1^{18} + 360\omega_3^2\omega_1^{16} + 3840\omega_1^{16} - 243\omega_3^4\omega_1^{14} - 21096\omega_3^2\omega_1^{14} - 60288\omega_1^{14} + \\
& 12960\omega_3^4\omega_1^{12} + 341100\omega_3^2\omega_1^{12} + 98560\omega_1^{12} - 195696\omega_3^4\omega_1^{10} - 697824\omega_3^2\omega_1^{10} + \\
& 1128960\omega_1^{10} + 364500\omega_3^4\omega_1^8 - 8339760\omega_3^2\omega_1^8 - 19906560\omega_1^8 + 4397328\omega_3^4\omega_1^6 + \\
& 66764736\omega_3^2\omega_1^6 + 90616320\omega_1^6 - 27818640\omega_3^4\omega_1^4 - 229139280\omega_3^2\omega_1^4 - \\
& 119439360\omega_1^4 + 106655616\omega_3^4\omega_1^2 + 229267584\omega_3^2\omega_1^2 + 53747712\omega_1^2 - \\
& 75582720\omega_3^4 - 75582720\omega_3^2) , \\
a_{13} &= \tfrac{1}{\delta_{13}} (-12\omega_3^2\omega_1^{22} + 19\omega_3^4\omega_1^{20} + 1384\omega_3^2\omega_1^{20} + 256\omega_1^{20} - 9\omega_3^6\omega_1^{18} - 2422\omega_3^4\omega_1^{18} - \\
& 55404\omega_3^2\omega_1^{18} - 29952\omega_1^{18} + 1233\omega_3^6\omega_1^{16} + 103915\omega_3^4\omega_1^{16} + 870080\omega_3^2\omega_1^{16} + \\
& 1108992\omega_1^{16} - 81\omega_3^8\omega_1^{14} - 56709\omega_3^6\omega_1^{14} - 1594980\omega_3^4\omega_1^{14} - 4398976\omega_3^2\omega_1^{14} - \\
& 13969408\omega_1^{14} + 6561\omega_3^8\omega_1^{12} + 910845\omega_3^6\omega_1^{12} + 3240656\omega_3^4\omega_1^{12} + 16711296\omega_3^2\omega_1^{12} + \\
& 33681408\omega_1^{12} - 139968\omega_3^8\omega_1^{10} - 979452\omega_3^6\omega_1^{10} + 26756208\omega_3^4\omega_1^{10} - \\
& 16932848\omega_3^2\omega_1^{10} + 7299072\omega_1^{10} + 180144\omega_3^8\omega_1^8 - 25675380\omega_3^6\omega_1^8 - \\
& 110852820\omega_3^4\omega_1^8 - 43948800\omega_3^2\omega_1^8 - 71663616\omega_1^8 + 4420656\omega_3^8\omega_1^6 + \\
& 83144880\omega_3^6\omega_1^6 + 188930880\omega_3^4\omega_1^6 + 35256384\omega_3^2\omega_1^6 + \\
& 35831808\omega_1^6 - 14568336\omega_3^8\omega_1^4 - 144050400\omega_3^6\omega_1^4 - 184162896\omega_3^4\omega_1^4 + \\
& 110108160\omega_3^2\omega_1^4 + 26034048\omega_3^8\omega_1^2 + 117573120\omega_3^6\omega_1^2 + 78102144\omega_3^4\omega_1^2 - \\
& 73903104\omega_3^2\omega_1^2 - 15116544\omega_3^8 - 30233088\omega_3^6 - 15116544\omega_3^4) ,
\end{aligned}$$

$$
\begin{aligned}
a_{14} &= \tfrac{1}{\delta_{14}}\,(-8\omega_1^{16} + 27\omega_3^2\omega_1^{14} + 368\omega_1^{14} - 711\omega_3^2\omega_1^{12} - 5860\omega_1^{12} + 15732\omega_3^2\omega_1^{10} + \\
&\quad 9676\omega_1^{10} - 17316\omega_3^2\omega_1^8 + 281664\omega_1^8 - 562140\omega_3^2\omega_1^6 - 1188144\omega_1^6 + 1533168\omega_3^2\omega_1^4 + \\
&\quad 1950480\omega_1^4 - 1411344\omega_3^2\omega_1^2 - 1458000\omega_1^2 + 419904\omega_3^2 + 419904\,, \\
a_{15} &= \tfrac{1}{\delta_{15}}\,(-7\omega_3^2\omega_1^{20} + 9\omega_3^4\omega_1^{18} + 786\omega_3^2\omega_1^{18} + 64\omega_1^{18} - 981\omega_3^4\omega_1^{16} - 31935\omega_3^2\omega_1^{16} - \\
&\quad 5952\omega_1^{16} + 81\omega_3^6\omega_1^{14} + 40545\omega_3^4\omega_1^{14} + 568528\omega_3^2\omega_1^{14} + 92928\omega_1^{14} - 7209\omega_3^6\omega_1^{12} - \\
&\quad 880677\omega_3^4\omega_1^{12} - 3781728\omega_3^2\omega_1^{12} + 3548672\omega_1^{12} + 269568\omega_3^6\omega_1^{10} + 10030716\omega_3^4\omega_1^{10} - \\
&\quad 19659744\omega_3^2\omega_1^{10} - 83248128\omega_1^{10} - 4420656\omega_3^6\omega_1^8 + 12733524\omega_3^4\omega_1^8 + \\
&\quad 427855284\omega_3^2\omega_1^8 + 255301632\omega_1^8 - 1621296\omega_3^6\omega_1^6 - 496408176\omega_3^4\omega_1^6 - \\
&\quad 1326103488\omega_3^2\omega_1^6 - 256794624\omega_1^6 + 164707344\omega_3^6\omega_1^4 + 1199035872\omega_3^4\omega_1^4 + \\
&\quad 1549970640\omega_3^2\omega_1^4 + 80621568\omega_1^4 - 294772608\omega_3^6\omega_1^2 - 997691904\omega_3^4\omega_1^2 - \\
&\quad 763385472\omega_3^2\omega_1^2 + 136048896\omega_3^6 + 272097792\omega_3^4 + 136048896\omega_3^2)\,, \\
a_{16} &= -\tfrac{\omega_1^8 - 80\omega_1^6 + 1348\omega_1^4 - 2880\omega_1^2 + 1296}{4\left(\omega_1^4 - 36\right)\left(\omega_1^4 - 40\omega_1^2 + 36\right)\omega_3^2}, \\
\gamma_1 &= -\tfrac{1}{\delta_{16}}\,(\omega_1^2 - 9)\,(3\omega_3^2\omega_1^{12} - 8\omega_1^{12} - 36\omega_3^2\omega_1^{10} + 364\omega_1^{10} - 1200\omega_3^2\omega_1^8 - 6992\omega_1^8 + \\
&\quad 3636\omega_3^2\omega_1^6 - 13488\omega_1^6 + 38016\omega_3^2\omega_1^4 + 92736\omega_1^4 - 84240\omega_3^2\omega_1^2 - 120528\omega_1^2 + \\
&\quad 46656\omega_3^2 + 46656)
\end{aligned}
$$

with

$$
\begin{aligned}
\delta_{11} &= 4096\omega_1^4\,(\omega_1^2 - 36)\,(\omega_1^2 - 9)\,(4\omega_1^2 - \omega_3^2)\,, \\
\delta_{12} &= 3072\omega_1^4\,(\omega_1^4 - 40\omega_1^2 + 144)\,(64\omega_1^4 - 180\omega_3^2\omega_1^2 + 81\omega_3^4)\,, \\
\delta_{13} &= 64\omega_1^3\,(\omega_1^2 - 36)\,(\omega_1^2 - 4)\,(\omega_1^4 - 40\omega_1^2 + 36)\,\omega_3\,(256\omega_1^6 - 784\omega_3^2\omega_1^4 + \\
&\quad 504\omega_3^4\omega_1^2 - 81\omega_3^6)\,, \\
\delta_{14} &= 36864\omega_1^4\,(\omega_1^4 - 5\omega_1^2 + 4)\,(4\omega_1^2 - 9\omega_3^2)\,, \\
\delta_{15} &= 192\omega_1^3\,(\omega_1^2 - 36)\,(\omega_1^2 - 4)\,(\omega_1^4 - 40\omega_1^2 + 36)\,\omega_3\,(64\omega_1^4 - 180\omega_3^2\omega_1^2 + 81\omega_3^4)\,, \\
\delta_{16} &= 1024\sqrt{3}\omega_1^4\sqrt{\omega_1^8 - 50\omega_1^6 + 553\omega_1^4 - 1800\omega_1^2 + 1296}\,(4\omega_1^2 - 9\omega_3^2)\,.
\end{aligned}
$$

In order to apply our Theorem 3.2 is necessary to verify the condition $3\sqrt{3}|\gamma_1| > |a_{11}+3a_{12}+9a_{14}|$. We verify that this inequality holds in the following two intervals:

$$1.09641 < C < 1.22754, \quad 1.68463 < C < 1.88687,$$

and therefore, by Theorem 3.2 the equilibrium point P_1 is unstable in the Liapunov sense in these intervals (see Figure 5.7).

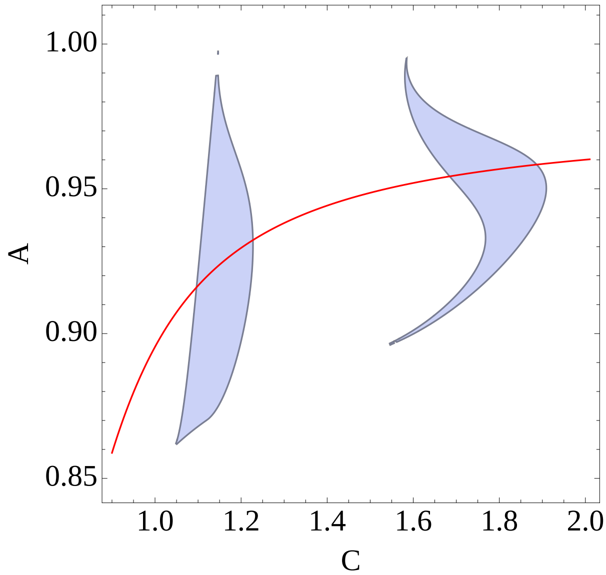


Figure 5.7: Representation of the three curves of instability of the equilibrium P_1 on the resonance curve defined by $\omega_1 - 3\omega_2 = 0$. The "island" regions represents the region defined by (C, A) such that $3\sqrt{3}|\gamma_1| > |a_{11} + 3a_{12} + 9a_{14}|$ inside the region of linear stability. Note that the curves of instability are intersection between the island regions and the curve of resonance.

For the case $k = (0, 1, 1)$, we compute that the Lie normalized Hamiltonian up to order four is

$$\begin{aligned}\mathcal{H} &= \omega_1 I_1 - \omega_3 I_2 + 3\omega_2 I_3 + a_{21} I_1^2 + a_{22} I_1 I_2 + a_{23} I_1 I_3 + a_{24} I_2^2 + \\ &\quad a_{25} I_2 I_3 + a_{26} I_3^2 + \gamma_2 I_2 I_3 \cos(2\varphi_2 + 2\varphi_3) + \cdots ,\end{aligned} \tag{5.5}$$

where

$$\begin{aligned}a_{21} &= \tfrac{1}{\delta_{21}} (3\omega_1^6\omega_3^{10} - 7\omega_1^4\omega_3^{10} + 20\omega_1^2\omega_3^{10} - 8\omega_1^8\omega_3^8 + 20\omega_1^6\omega_3^8 - 48\omega_1^4\omega_3^8 - 12\omega_1^2\omega_3^8 - \\ &\quad 64\omega_3^8 + 44\omega_1^8\omega_3^6 - 84\omega_1^6\omega_3^6 + 20\omega_1^4\omega_3^6 + 148\omega_1^2\omega_3^6 + 112\omega_3^6 - 64\omega_1^8\omega_3^4 - 64\omega_1^6\omega_3^4 + \\ &\quad 368\omega_1^4\omega_3^4 - 480\omega_1^2\omega_3^4 + 32\omega_3^4 + 64\omega_1^8\omega_3^2 + 192\omega_1^6\omega_3^2 - 464\omega_1^4\omega_3^2 + 416\omega_1^2\omega_3^2 - \\ &\quad 144\omega_3^2 - 256\omega_1^6 + 320\omega_1^4 - 128\omega_1^2 + 64) ,\end{aligned}$$

$$
\begin{aligned}
a_{22} &= \tfrac{1}{\delta_{22}} \left(-\omega_3^8\omega_1^{10} + 6\omega_3^6\omega_1^{10} - 6\omega_3^4\omega_1^{10} - 8\omega_3^2\omega_1^{10} + 6\omega_3^{10}\omega_1^8 - 28\omega_3^8\omega_1^8 - 10\omega_3^6\omega_1^8 + \right.\\
&\quad 106\omega_3^4\omega_1^8 + 16\omega_3^2\omega_1^8 - 50\omega_3^{10}\omega_1^6 + 260\omega_3^8\omega_1^6 - 246\omega_3^6\omega_1^6 - 32\omega_3^4\omega_1^6 - \\
&\quad 384\omega_3^2\omega_1^6 + 128\omega_1^6 + 88\omega_3^{10}\omega_1^4 - 224\omega_3^8\omega_1^4 - 616\omega_3^6\omega_1^4 + 1320\omega_3^4\omega_1^4 + 264\omega_3^2\omega_1^4 - \\
&\quad 256\omega_1^4 - 8\omega_3^{10}\omega_1^2 - 360\omega_3^8\omega_1^2 + 1608\omega_3^6\omega_1^2 - 2584\omega_3^4\omega_1^2 + 640\omega_3^2\omega_1^2 + 128\omega_1^2 + \\
&\quad \left. 128\omega_3^8 - 256\omega_3^6 + 512\omega_3^4 - 384\omega_3^2\right), \\
a_{23} &= \tfrac{1}{\delta_{23}} \left(-\omega_3^{10}\omega_1^{12} + 12\omega_3^8\omega_1^{12} - 44\omega_3^6\omega_1^{12} + 16\omega_3^4\omega_1^{12} + 192\omega_3^2\omega_1^{12} - 256\omega_1^{12} - \right.\\
&\quad \omega_3^{12}\omega_1^{10} + 17\omega_3^{10}\omega_1^{10} - 125\omega_3^8\omega_1^{10} + 456\omega_3^6\omega_1^{10} - 816\omega_3^4\omega_1^{10} + 448\omega_3^2\omega_1^{10} + \\
&\quad 768\omega_1^{10} + 5\omega_3^{12}\omega_1^8 - 4\omega_3^{10}\omega_1^8 - 7\omega_3^8\omega_1^8 - 212\omega_3^6\omega_1^8 + 544\omega_3^4\omega_1^8 - 2432\omega_3^2\omega_1^8 - \\
&\quad 8\omega_3^{12}\omega_1^6 - 284\omega_3^{10}\omega_1^6 + 1220\omega_3^8\omega_1^6 - 304\omega_3^6\omega_1^6 + 2064\omega_3^4\omega_1^6 + 320\omega_3^2\omega_1^6 - \\
&\quad 1280\omega_1^6 + 68\omega_3^{12}\omega_1^4 + 276\omega_3^{10}\omega_1^4 - 1348\omega_3^8\omega_1^4 - 4564\omega_3^6\omega_1^4 + 3616\omega_3^4\omega_1^4 + \\
&\quad 896\omega_3^2\omega_1^4 + 768\omega_1^4 - 208\omega_3^{12}\omega_1^2 + 576\omega_3^{10}\omega_1^2 + 1696\omega_3^8\omega_1^2 - 2432\omega_3^6\omega_1^2 + \\
&\quad \left. 240\omega_3^4\omega_1^2 + 128\omega_3^2\omega_1^2 - 256\omega_3^{10} + 1792\omega_3^8 - 4096\omega_3^6 + 3840\omega_3^4 - 1280\omega_3^2\right), \\
a_{24} &= \tfrac{1}{\delta_{24}} \left(-5\omega_3^8\omega_1^8 + 29\omega_3^6\omega_1^8 - 40\omega_3^4\omega_1^8 + 16\omega_3^2\omega_1^8 + 16\omega_1^8 + 28\omega_3^8\omega_1^6 - 92\omega_3^6\omega_1^6 - \right.\\
&\quad 48\omega_3^4\omega_1^6 + 112\omega_3^2\omega_1^6 - 112\omega_1^6 - 44\omega_3^8\omega_1^4 - 8\omega_3^6\omega_1^4 + 388\omega_3^4\omega_1^4 - 336\omega_3^2\omega_1^4 + \\
&\quad 240\omega_1^4 + 48\omega_3^8\omega_1^2 + 128\omega_3^6\omega_1^2 - 448\omega_3^4\omega_1^2 + 272\omega_3^2\omega_1^2 - 208\omega_1^2 - 192\omega_3^6 + \\
&\quad \left. 256\omega_3^4 - 64\omega_3^2 + 64\right), \\
a_{25} &= \tfrac{1}{\delta_{25}} \left(-2\omega_1^{10}\omega_3^{12} + 10\omega_1^8\omega_3^{12} - 8\omega_1^6\omega_3^{12} + 24\omega_1^4\omega_3^{12} - 96\omega_1^2\omega_3^{12} + 24\omega_1^{10}\omega_3^{10} - \right.\\
&\quad 70\omega_1^8\omega_3^{10} - 28\omega_1^6\omega_3^{10} - 376\omega_1^4\omega_3^{10} + 1152\omega_1^2\omega_3^{10} - 103\omega_1^{10}\omega_3^8 + 68\omega_1^8\omega_3^8 + \\
&\quad 636\omega_1^6\omega_3^8 + 1096\omega_1^4\omega_3^8 - 3776\omega_1^2\omega_3^8 + 185\omega_1^{10}\omega_3^6 + 440\omega_1^8\omega_3^6 - 1228\omega_1^6\omega_3^6 - \\
&\quad 1080\omega_1^4\omega_3^6 + 4992\omega_1^2\omega_3^6 - 1536\omega_3^6 - 152\omega_1^{10}\omega_3^4 - 1632\omega_1^8\omega_3^4 + 148\omega_1^6\omega_3^4 + \\
&\quad 2688\omega_1^4\omega_3^4 - 4448\omega_1^2\omega_3^4 + 3072\omega_3^4 + 272\omega_1^{10}\omega_3^2 + 2496\omega_1^8\omega_3^2 - 5664\omega_1^6\omega_3^2 + \\
&\quad 2512\omega_1^4\omega_3^2 + 1920\omega_1^2\omega_3^2 - 1536\omega_3^2 - 512\omega_1^{10} + 1280\omega_1^8 - 768\omega_1^6 - 256\omega_1^4 + \\
&\quad \left. 256\omega_1^2\right), \\
a_{26} &= -\frac{\left(\omega_3^2-4\right)^2\omega_1^4+4\left(-2\omega_3^4+\omega_3^2-8\right)\omega_1^2+16\left(\omega_3^2-1\right)^2}{4\omega_3^2\left(\omega_1^2\omega_3^2-4\right)\left(\left(\omega_3^2-4\right)\omega_1^2-4\omega_3^2+4\right)}, \\
\gamma_2 &= \tfrac{1}{\delta_{26}} \left(-4\omega_1^6\omega_3^{10} + 20\omega_1^4\omega_3^{10} - 16\omega_1^2\omega_3^{10} + 5\omega_1^8\omega_3^8 + 12\omega_1^6\omega_3^8 - 108\omega_1^4\omega_3^8 + \right.\\
&\quad 64\omega_1^2\omega_3^8 - 37\omega_1^8\omega_3^6 + 204\omega_1^4\omega_3^6 - 32\omega_1^2\omega_3^6 + 72\omega_1^8\omega_3^4 + 192\omega_1^6\omega_3^4 - 308\omega_1^4\omega_3^4 - \\
&\quad 320\omega_1^2\omega_3^4 + 256\omega_3^4 - 80\omega_1^8\omega_3^2 - 256\omega_1^6\omega_3^2 - 480\omega_1^4\omega_3^2 + 1328\omega_1^2\omega_3^2 - 512\omega_3^2 + \\
&\quad \left. 256\omega_1^8 - 1024\omega_1^6 + 1536\omega_1^4 - 1024\omega_1^2 + 256\right).
\end{aligned}
$$

with

$$
\begin{aligned}
\delta_{21} &= 64\left(\omega_1^2-\omega_3^2\right)^2\left(4\omega_1^2-\omega_3^2\right)\left(\omega_3^4-5\omega_3^2+4\right),\\
\delta_{22} &= 16\omega_1\left(\omega_1^2-4\right)\omega_3\left(\omega_1^2-4\omega_3^2\right)\left(\omega_1^2-\omega_3^2\right)^2\left(\omega_3^2-4\right),\\
\delta_{23} &= 8\omega_1\left(\omega_1^2-4\right)\left(\omega_1^2-\omega_3^2\right)\omega_3\left(\omega_3^2-4\right)\left(4\omega_1^4-17\omega_3^2\omega_1^2+4\omega_3^4\right)\times\\
&\quad\left(\left(\omega_3^2-4\right)\omega_1^2-4\omega_3^2+4\right),\\
\delta_{24} &= 192\left(\omega_1^4-5\omega_1^2+4\right)\omega_3^2\left(\omega_1^2-\omega_3^2\right)^2,\\
\delta_{25} &= 24\omega_1^2\left(\omega_1^2-4\right)\left(\omega_1^2-\omega_3^2\right)\omega_3^2\left(\omega_1^2-4\omega_3^2\right)\left(\omega_3^2-4\right)\left(\left(\omega_3^2-4\right)\omega_1^2-4\omega_3^2+4\right),\\
\delta_{26} &= 16\omega_1^2\left(\omega_1^2-4\right)\left(\omega_1^2-\omega_3^2\right)\omega_3^2\left(\omega_3^2-4\right)\left(\left(\omega_3^2-4\right)\omega_1^2-4\omega_3^2+4\right).
\end{aligned}
$$

In order to apply our Theorem 3.2 it is necessary to verify the condition $4|\gamma_2| > 4|a_{24}+a_{25}+a_{26}|$, that is, in the interval $1 < C < 1.03398$, and therefore, by Theorem 3.2 the equilibrium point P_1 is unstable in the Liapunov sense in this interval (see Figure 5.8).

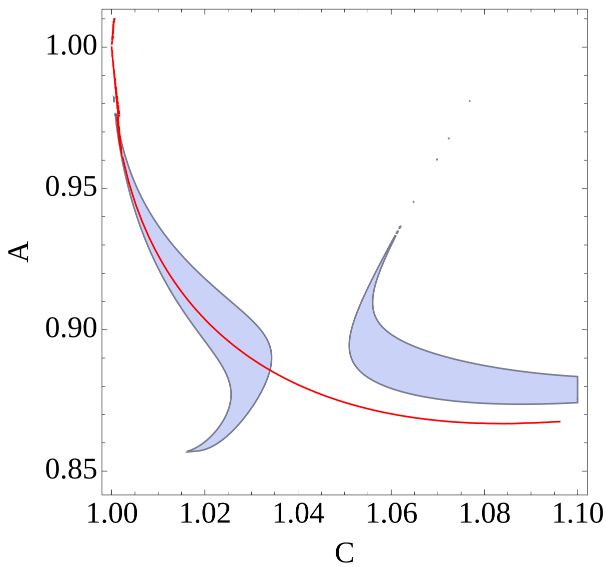


Figure 5.8: Representation of the curve of instability of the equilibrium P_1 on the resonance curve defined by $\omega_2-\omega_3=0$. The "island" regions represents the region defined by (C, A) such that $3\sqrt{3}|\gamma_2| > |a_{21}+3a_{22}+9a_{24}|$ inside the region of linear stability. Note that the curve of instability is the intersection between the island regions and the curve of resonance.

On the other hand, for the cases $k=(1,2,1)$, $k=(0,3,1)$ and $k=(0,1,3)$, we have that the quartic part associated to the Lie normalized Hamiltonian does not have resonant terms, so we can not apply Theorem 3.2.

5.4 Instability at the point P_3 in the case of multiple resonance

We determine that in the region II there are exactly seven resonance curves of order 3, which are characterized by $c_{13} : 2\omega_1 - \omega_3 = 0$, $c_{23} : \omega_1 - 2\omega_2 = 0$, $c_{33} : \omega_1 - \omega_2 - \omega_3 = 0$, $c_{43} : \omega_1 - 2\omega_3 = 0$, $c_{53} : \omega_1 + \omega_2 - \omega_3 = 0$, $c_{63} : 2\omega_2 - \omega_3 = 0$ and $c_{73} : \omega_2 - 2\omega_3 = 0$. As we saw in Theorem 3.4 a necessary condition to have instability in the case of multiple resonance is that at least one of the resonance vectors has all the components of the resonance vector with the same sign. So, we only need to analyze the situations shown in Table 5.12.

Case	k_1	k_2	Point (C, A)	Curves of resonance
1	$(0, 2, 1)$	$(1, 1, -1)$	$(1.19633, 1.09188)$	$\omega_3 - 2\omega_2 = 0,\ \omega_1 - \omega_2 - \omega_3 = 0$
2	$(0, 2, 1)$	$(1, 0, -2)$	$(1.14105, 1.078)$	$\omega_3 - 2\omega_2 = 0,\ \omega_1 - 2\omega_3 = 0$
3	$(0, 2, 1)$	$(1, 2, 0)$	$(1.31629, 1.11029)$	$\omega_3 - 2\omega_2 = 0,\ \omega_1 - 2\omega_2 = 0$

Table 5.12: Characterization of the resonances of order three.

For the previous cases, the Hamiltonian in its Lie normal form (for more details see [55]) up to order three is $\mathcal{H} = \omega_1 I_1 - \omega_2 I_2 + \omega_3 I_3 + \mathcal{H}_3$, whose cubic part is given in Table 5.13.

Case	$\mathcal{H}_3$
1	$0.808061\sqrt{I_1 I_2 I_3}\cos(\varphi_1 + \varphi_2 - \varphi_3) - 0.541235 I_2\sqrt{I_3}\cos(2\varphi_2 + \varphi_3)$
2	$-0.426858 I_2\sqrt{I_3}\cos(2\varphi_2 + \varphi_3)$
3	$-0.87278 I_2\sqrt{I_3}\cos(2\varphi_2 + \varphi_3)$

Table 5.13: Lie normal form of the terms of order three.

The invariant ray solution of the form $I_j = c_j b(t)$ and $\theta_j = \theta_j^0$ defined in (3.30), for the previous cases is given by Table 5.14.

Case	c_1	c_2	c_3	θ_1^0	θ_2^0
1	6.85089	24.0437	1.74549	$\frac{\pi}{2}$	$-\frac{\pi}{2}$
2	10.9765	5.48824	1.5708	$\frac{\pi}{2}$	
3	1	2.62555	1.31277	$\frac{\pi}{2}$	

Table 5.14: Invariant ray solution $I_1 = c_1 b(t), I_2 = c_2 b(t), I_3 = c_3 b(t), \theta_1 = \theta_1^0, \theta_2 = \theta_2^0$, in the cases of resonance multiple of order three.

Thus, in the cases that appear in the Table 5.12 the matrix $B_{\nu\zeta}$ in (3.50) is given by Table 5.15.

Case	1	2	3
$B_{\nu\zeta}$	$\begin{pmatrix} 0.819983 & -1.32935 & 0 & 0 \\ 4.46521 & -4.10492 & 0 & 0 \\ 0 & 0 & -6.35503 & 3.35503 \\ 0 & 0 & -5.63997 & 2.63997 \end{pmatrix}$	$\begin{pmatrix} -1. & -0.471405 & 0 & 0 \\ 0. & -2. & 0 & 0 \\ 0 & 0 & -3. & 0. \\ 0 & 0 & -2. & 0. \end{pmatrix}$	$\begin{pmatrix} -1. & -0.471405 & 0 & 0 \\ 0. & -2. & 0 & 0 \\ 0 & 0 & -3. & 0. \\ 0 & 0 & -2. & 0. \end{pmatrix}$

Table 5.15: Matrix $B_{\nu\zeta}$ in (3.50) for the cases of multiple resonance of order three.

Then just by choosing $\gamma = 0$, hence by Theorem 3.4 the null solution of the complete Hamiltonian system associated with the Hamiltonian (5.1) is unstable in the Liapunov sense.

Next, we compute that in the region II there are exactly twenty four resonance curves of order 5, which are characterized by $c_{15} : 4\omega_1 - \omega_3 = 0$, $c_{25} : 3\omega_1 - \omega_2 - \omega_3 = 0$, $c_{35} : 3\omega_1 - 2\omega_3 = 0$, $c_{45} : 3\omega_1 + \omega_2 - \omega_3 = 0$, $c_{55} : 2\omega_1 - 3\omega_2 = 0$, $c_{65} : 2\omega_1 - 2\omega_2 - \omega_3 = 0$, $c_{75} : 2\omega_1 - \omega_2 - 2\omega_3 = 0$, $c_{85} : 2\omega_1 - 3\omega_3 = 0$, $c_{95} : 2\omega_1 + \omega_2 - 2\omega_3 = 0$, $c_{105} : 2\omega_1 + 2\omega_2 - \omega_3 = 0$, $c_{115} : \omega_1 - 4\omega_2 = 0$, $c_{125} : \omega_1 - 3\omega_2 + \omega_3 = 0$, $c_{135} : \omega_1 - 3\omega_2 - \omega_3 = 0$, $c_{145} : \omega_1 - 2\omega_2 + 2\omega_3 = 0$, $c_{155} : \omega_1 - 2\omega_2 - 2\omega_3 = 0$, $c_{165} : \omega_1 - \omega_2 - 3\omega_3 = 0$, $c_{175} : \omega_1 - 4\omega_3 = 0$, $c_{185} : \omega_1 + \omega_2 - 3\omega_3 = 0$, $c_{195} : \omega_1 + 2\omega_2 - 2\omega_3 = 0$, $c_{205} : \omega_1 + 3\omega_2 - \omega_3 = 0$, $c_{215} : \omega_3 - 4\omega_2 = 0$, $c_{225} : 2\omega_3 - 3\omega_2 = 0$, $c_{235} : 3\omega_3 - 2\omega_2 = 0$, $c_{24} : 4\omega_3 - \omega_2 = 0$.
There are 45 cases of multiple resonance of order five, in which at least one of the vectors of resonance has all the components of the same sign, but only in 5 of these cases it is possible to apply our theorem, which appear in the Table 5.16.

Case	k_1	k_2	Point (C, A)	Curves of resonance
1	$(0, 4, 1)$	$(1, 1, -3)$	$(1.05431, 1.02515)$	$\omega_3 - 4\omega_2 = 0,\ \omega_1 - \omega_2 - 3\omega_3 = 0$
2	$(0, 4, 1)$	$(1, -1, -3)$	$(1.06825, 1.02788)$	$\omega_3 - 4\omega_2 = 0,\ \omega_1 + \omega_2 - 3\omega_3 = 0$
3	$(0, 2, 3)$	$(1, 3, -1)$	$(1.1073, 1.09973)$	$3\omega_3 - 2\omega_2 = 0,\ \omega_1 - 3\omega_2 - \omega_3 = 0$
4	$(1, 3, 1)$	$(2, 0, -3)$	$(1.21369, 1.13621)$	$\omega_1 - 3\omega_2 + \omega_3 = 0,\ 2\omega_1 - 3\omega_3 = 0$
5	$(0, 4, 1)$	$(3, 1, -1)$	$(2.41255, 1.07997)$	$\omega_3 - 4\omega_2 = 0,\ 3\omega_1 - \omega_2 - \omega_3 = 0$

Table 5.16: Characterization of the resonances of order five.

For the previous cases, the Hamiltonian (5.1) in its Lie normal form up to order five is $\mathcal{H} = \omega_1 I_1 - \omega_2 I_2 + \omega_3 I_3 + \mathcal{H}_5$, here it is verified that $\mathcal{H}_4 \equiv 0$ and the terms of order five are given in Table 5.17.

Case	$\mathcal{H}_5$
1	$11.4283\left(0.0712796\sqrt{I_1 I_2} I_3^{3/2}\cos\left(\varphi_1+\varphi_2-3\varphi_3\right)+0.0646402 I_2^2\sqrt{I_3}\cos\left(4\varphi_2+\varphi_3\right)\right)$
2	$8.25725\left(0.0867938 I_2^2\sqrt{I_3}\cos\left(4\varphi_2+\varphi_3\right)-0.136403\sqrt{I_1 I_2} I_3^{3/2}\cos\left(\varphi_1-\varphi_2-3\varphi_3\right)\right)$
3	$44.0026\left(0.0153433\sqrt{I_1 I_3} I_2^{3/2}\cos\left(\varphi_1+3\varphi_2-\varphi_3\right)+0.542985 I_3^{3/2} I_2\cos\left(2\varphi_2+3\varphi_3\right)\right)$
4	$6.19492\left(-6.70509\sqrt{I_1 I_3} I_2^{3/2}\cos\left(\varphi_1+3\varphi_2+\varphi_3\right)-7.13285 I_1 I_3^{3/2}\cos\left(2\varphi_1-3\varphi_3\right)\right)$
5	$0.25014\left(66.292 I_1^{3/2}\sqrt{I_2 I_3}\cos\left(3\varphi_1+\varphi_2-\varphi_3\right)-6.17557 I_2^2\sqrt{I_3}\cos\left(4\varphi_2+\varphi_3\right)\right)$

Table 5.17: Lie normal form of the terms of order five.

The invariant ray solution of the form $I_j = c_j b(t)$ and $\theta_j = \theta_j^0$ as was defined in (3.30), for the previous cases is given by Table 5.4.

Case	c_1	c_2	c_3	θ_1^0	θ_2^0
1	0.017279	4.10553	0.970226	$-\frac{\pi}{2}$	$-\frac{\pi}{2}$
2	0.0347708	4.27686	0.973594	$-\frac{\pi}{2}$	$\frac{\pi}{2}$
3	0.0000235488	0.265397	0.397966	$-\frac{\pi}{2}$	$-\frac{\pi}{2}$
4	0.203912	0.480595	0.0946277	$\frac{\pi}{2}$	$\frac{\pi}{2}$
5	0.0387255	2.48407	0.604881	$\frac{\pi}{2}$	$-\frac{\pi}{2}$

Table 5.18: Invariant ray solution $I_1 = c_1 b(t), I_2 = c_2 b(t), I_3 = c_3 b(t), \theta_1 = \theta_1^0, \theta_2 = \theta_2^0$, in the cases of resonance multiple of order five.

Thus, in the cases that appear in the Table 5.16 the matrix $B_{\nu\zeta}$ in (3.50) is given by Table 5.19.

Case	Matrix $B_{\nu\zeta}$	Case	Matrix $B_{\nu\zeta}$
1	$b_{11}=-0.939772, b_{12}=-2.36466, b_{21}=0.136691, b_{22}=-2.03724,$ $b_{33}=-5.03659, b_{34}=0.0365929, b_{43}=-4.15607, b_{44}=-0.843926$	4	$b_{11}=-1.67014, b_{12}=-0.341962, b_{21}=0.244988, b_{22}=-1.65353,$ $b_{33}=-5.47856, b_{34}=0.478556, b_{43}=-6.65004, b_{44}=1.65004$
2	$b_{11}=-0.865516, b_{12}=-3.27266, b_{21}=0.256539, b_{22}=-1.91797,$ $b_{33}=-5.13966, b_{34}=0.139662, b_{43}=-4.32955, b_{44}=-0.670445$	5	$b_{11}=0.995762, b_{12}=-1.42563, b_{21}=0.0240803, b_{22}=-2.03981,$ $b_{33}=-5.00055, b_{34}=0.000554485, b_{43}=-2.01614, b_{44}=-2.98386$
3	$b_{11}=-1.00006, b_{12}=1.41401, b_{21}=0.000428784, b_{22}=-1.99963,$ $b_{33}=-4.99965, b_{34}=-0.000354864, b_{43}=-3.99926, b_{44}=-1.00074$	$B_{\nu\zeta}$	$\begin{pmatrix} b_{11} & b_{12} & 0 & 0 \\ b_{21} & b_{22} & 0 & 0 \\ 0 & 0 & b_{33} & b_{34} \\ 0 & 0 & b_{43} & b_{44} \end{pmatrix}$

Table 5.19: Matrix $B_{\nu\zeta}$ in (3.50) for the cases of multiple resonance of order five.

According to (3.52) we just choose $\gamma = 0$, hence by Theorem 3.4 the null solution of the complete Hamiltonian system associated with the Hamiltonian (5.1) is unstable in the Liapunov sense.

5.5 Nekhoroshev stability

In this section we are going to study the Nekhoroshev stability and estimates of exponential time of the equilibrium point P_3. The analysis of Nekhoroshev stability will depend of the type steepness, namely, quasi-convexity, direccional quasi-convexity.

We normalize the Hamiltonian without resonance to order four

$$\mathcal{H} = \mathcal{H}(I) = h_2(I) + h_4(I) + \cdots, \tag{5.6}$$

where $h_4(I) = \frac{1}{2} IAI = a_{200}I_1^2 + a_{020}I_2^2 + a_{002}I_3^2 + a_{110}I_1I_2 + a_{011}I_2I_3 + a_{101}I_1I_3$, with

$$\begin{aligned}
a_{200} &= \frac{f_2\left[4\left(-4g_1^3g_2-4g_1^2g_3\omega_1^2+g_6\omega_1^4+g_5\omega_1^6+g_2g_4\omega_1^8\omega_2^2\right)-h_2\left(4g_1^2g_7+g_9\omega_1^2+g_2g_8\omega_1^4\right)\omega_3^2\right]}{192f_{10}g_1h_1^2h_2},\\
a_{020} &= \frac{g_2\left[4\left(-4f_1^3f_2-4f_1^2f_3\omega_2^2+f_6\omega_2^4+f_5\omega_2^6+f_2f_4\omega_1^8\omega_2^2\right)-h_2\left(4f_1^2f_7+f_9\omega_2^2+f_2f_8\omega_2^4\right)\omega_3^2\right]}{192g_{10}f_1h_1^2h_2},\\
a_{002} &= -\frac{A}{4},\\
a_{110} &= \frac{8t_4\omega_1^2+2t_3\omega_1^4-2t_2\omega_1^6+t_1\omega_1^8-128g_1t_{19}\omega_2^2-2t_5\omega_1^{10}\omega_2^2+t_{18}\omega_1^{12}\omega_2^2}{48h_1^2h_2h_3\omega_1\omega_2},\\
a_{011} &= \frac{64g_1t_{17}-16t_{16}\omega_1^2+4t_{14}\omega_1^4+t_{15}\omega_1^6+g_2t_{13}\omega_1^8+g_2^2t_{12}\omega_1^{10}}{24g_{10}h_1h_2h_3\omega_2(4+g_2\omega_1^2-4\omega_2^2)\omega_3},\\
a_{101} &= \frac{-16g_1t_{11}\omega_1^2+4t_8\omega_1^4-t_6\omega_1^6+t_7\omega_1^8-g_2t_9\omega_1^{10}+g_2t_{10}\omega_1^{12}+g_2^3t_{20}\omega_1^{14}-256g_1^2t_{21}\omega_3^2}{24f_{10}h_1h_2h_3\omega_1(4+g_2\omega_1^2-4\omega_2^2)\omega_3},
\end{aligned}$$

where

$$\begin{aligned}
f_1 &= \omega_1^2 - 1, & g_1 &= \omega_2^2 - 1,\\
f_2 &= \omega_1^2 - 4, & g_2 &= \omega_2^2 - 4,\\
f_3 &= \omega_1^4 + 14\omega_1^2 - 8, & g_3 &= \omega_2^4 + 14\omega_2^2 - 8,\\
f_4 &= 2\omega_1^4 - 3\omega_1^2 + 4, & g_4 &= 2\omega_2^4 - 3\omega_2^2 + 4,\\
f_5 &= -9\omega_1^8 + 26\omega_1^6 + 12\omega_1^4 - 48\omega_1^2 + 64, & g_5 &= -9\omega_2^8 + 26\omega_2^6 + 12\omega_2^4 - 48\omega_2^2 + 64,\\
f_6 &= 15\omega_1^8 + 5\omega_1^6 - 108\omega_1^4 + 132\omega_1^2 - 80, & g_6 &= 15\omega_2^8 + 5\omega_2^6 - 108\omega_2^4 + 132\omega_2^2 - 80,\\
f_7 &= 5\omega_1^2 - 4, & g_7 &= 5\omega_2^2 - 4,\\
f_8 &= 3\omega_1^4 - 4\omega_1^2 + 4, & g_8 &= 3\omega_2^4 - 4\omega_2^2 + 4,\\
f_9 &= -7\omega_1^6 + 24\omega_1^4 - 24\omega_1^2 + 16, & g_9 &= -7\omega_2^6 + 24\omega_2^4 - 24\omega_2^2 + 16,\\
f_{10} &= 4\omega_1^2 - \omega_3^2, & g_{10} &= 4\omega_2^2 - \omega_3^2,\\
h_1 &= \omega_1^2 - \omega_2^2, & h_2 &= \omega_1^2\omega_2^2 - 4,\\
h_3 &= \omega_1^4 - 2\left(\omega_2^2 + \omega_3^2\right)\omega_1^2 + \left(\omega_2^2 - \omega_3^2\right)^2. & &
\end{aligned}$$

and

$$
\begin{aligned}
t_1 &= \omega_2^{12} + \left(6 - 4\omega_3^2\right)\omega_2^{10} + \left(3\omega_3^4 + 60\omega_3^2 - 88\right)\omega_2^8 - 2\left(8\omega_3^4 + 97\omega_3^2 - 45\right)\omega_2^6 + \\
&\quad 2\left(11\omega_3^4 + 78\omega_3^2 - 20\right)\omega_2^4 + 24\left(3\omega_3^2 + 16\right)\omega_2^2 - 128, \\
t_2 &= 3\omega_2^{12} - \left(11\omega_3^2 + 19\right)\omega_2^{10} + \left(8\omega_3^4 + 97\omega_3^2 - 45\right)\omega_2^8 - 2\left(17\omega_3^4 + 46\omega_3^2 + 28\right)\omega_2^6 + \\
&\quad \left(25\omega_3^4 - 280\omega_3^2 + 192\right)\omega_2^4 + 4\left(7\omega_3^4 + 117\omega_3^2 + 65\right)\omega_2^2 - 128\left(\omega_3^2 + 1\right), \\
t_3 &= 3\omega_2^{12} - 7\left(2\omega_3^2 + 7\right)\omega_2^{10} + \left(11\omega_3^4 + 78\omega_3^2 - 20\right)\omega_2^8 + \left(-25\omega_3^4 + 280\omega_3^2 - 192\right)\omega_2^6 - \\
&\quad 8\left(10\omega_3^4 + 103\omega_3^2 - 81\right)\omega_2^4 + 4\left(53\omega_3^4 + 177\omega_3^2 - 32\right)\omega_2^2 - 64\left(\omega_3^4 + 3\omega_3^2 + 1\right), \\
t_4 &= \omega_2^{12} - \left(\omega_3^2 + 2\right)\omega_2^{10} + \left(9\omega_3^2 + 48\right)\omega_2^8 - \left(7\omega_3^4 + 117\omega_3^2 + 65\right)\omega_2^6 + \\
&\quad \left(53\omega_3^4 + 177\omega_3^2 - 32\right)\omega_2^4 - 16\left(5\omega_3^4 + 3\omega_3^2 - 2\right)\omega_2^2 + 16\left(\omega_3^4 + \omega_3^2\right), \\
t_5 &= \omega_2^8 + \left(2\omega_3^2 - 3\right)\omega_2^6 - \left(11\omega_3^2 + 19\right)\omega_2^4 + 7\left(2\omega_3^2 + 7\right)\omega_2^2 + 4\left(\omega_3^2 + 2\right), \\
t_6 &= \left(37\omega_3^4 + 293\omega_3^2 + 48\right)\omega_2^{10} - 2\left(23\omega_3^6 + 271\omega_3^4 - 14\omega_3^2 + 368\right)\omega_2^8 + \\
&\quad \left(9\omega_3^8 + 361\omega_3^6 + 608\omega_3^4 - 3088\omega_3^2 - 768\right)\omega_2^6 + \\
&\quad 4\omega_3^2\left(-11\omega_3^6 - 106\omega_3^4 + 843\omega_3^2 + 391\right)\omega_2^4 + 16\left(\omega_3^8 - 60\omega_3^6 - 399\omega_3^4 + 74\omega_3^2 + 16\right)\omega_2^2 + \\
&\quad 64\left(\omega_3^8 + 16\omega_3^6 + 38\omega_3^4 + 10\omega_3^2 + 12\right), \\
t_7 &= 2\left(5\omega_3^4 + 72\omega_3^2 + 18\right)\omega_2^{10} - \left(11\omega_3^6 + 247\omega_3^4 + 478\omega_3^2 + 96\right)\omega_2^8 + \\
&\quad \left(\omega_3^8 + 119\omega_3^6 + 1054\omega_3^4 - 592\omega_3^2 - 128\right)\omega_2^6 - \\
&\quad 4\left(2\omega_3^8 + 86\omega_3^6 + 157\omega_3^4 - 516\omega_3^2 + 336\right)\omega_2^4 + \\
&\quad 16\left(\omega_3^8 + 6\omega_3^6 - 122\omega_3^4 - 113\omega_3^2 - 72\right)\omega_2^2 + 64\left(5\omega_3^6 + 17\omega_3^4 + 13\omega_3^2 + 20\right), \\
t_8 &= 2\left(9\omega_3^4 + 35\omega_3^2 - 40\right)\omega_2^{10} - \left(27\omega_3^6 + 115\omega_3^4 - 268\omega_3^2 + 336\right)\omega_2^8 + \\
&\quad \left(9\omega_3^8 + 121\omega_3^6 - 469\omega_3^4 - 865\omega_3^2 + 864\right)\omega_2^6 + \\
&\quad \left(-24\omega_3^8 + 161\omega_3^6 + 1525\omega_3^4 + 752\omega_3^2 - 832\right)\omega_2^4 - \\
&\quad 4\left(6\omega_3^8 + 199\omega_3^6 + 376\omega_3^4 - 41\omega_3^2 - 96\right)\omega_2^2 + 16\omega_3^2\left(3\omega_3^6 + 31\omega_3^4 + 38\omega_3^2 - 26\right), \\
t_9 &= \left(\omega_3^4 + 34\omega_3^2 + 4\right)\omega_2^8 - \left(\omega_3^6 + 39\omega_3^4 + 127\omega_3^2 - 40\right)\omega_2^6 + \\
&\quad \left(10\omega_3^6 + 163\omega_3^4 + 133\omega_3^2 + 16\right)\omega_2^4 + 4\left(-6\omega_3^6 - 29\omega_3^4 + 12\omega_3^2 + 120\right)\omega_2^2 + \\
&\quad 16\omega_3^2\left(8 - 3\omega_3^2\right), \\
t_{10} &= \omega_2^2\left(-2\omega_2^4 + 17\omega_2^2 - 36\right)\omega_3^4 + \left(\omega_2^2 - 4\right)^2\left(3\omega_2^4 - 6\omega_2^2 + 5\right)\omega_3^2 + \\
&\quad 4\left(2\omega_2^6 - 7\omega_2^4 + 20\omega_2^2 + 48\right), \\
t_{11} &= \left(4\omega_2^4 + 5\omega_2^2 - 24\right)\omega_3^8 - \left(9\omega_2^6 + 18\omega_2^4 - 130\omega_2^2 + 112\right)\omega_3^6 + \\
&\quad \left(5\omega_2^8 + 16\omega_2^6 - 191\omega_2^4 + 293\omega_2^2 - 72\right)\omega_3^4 + \\
&\quad \left(-3\omega_2^8 + 117\omega_2^6 - 181\omega_2^4 - 40\omega_2^2 + 80\right)\omega_3^2 - 48\omega_2^4\left(\omega_2^2 - 1\right)^2, \\
t_{12} &= 3\omega_3^2\omega_2^8 - \left(\omega_3^4 + 10\omega_3^2 + 4\right)\omega_2^6 + 2\left(\omega_3^4 + 8\omega_3^2 + 2\right)\omega_2^4 + \left(-5\omega_3^4 - 5\omega_3^2 + 48\right)\omega_2^2 - \\
&\quad 16\omega_3^2,
\end{aligned}
$$

$$
\begin{aligned}
t_{13} &= \omega_3^2\omega_2^{12} - 2\left(\omega_3^4 + 19\omega_3^2 - 4\right)\omega_2^{10} + \left(\omega_3^6 + 35\omega_3^4 + 111\omega_3^2 + 8\right)\omega_2^8 - \\
&\left(7\omega_3^6 + 107\omega_3^4 + 34\omega_3^2 + 64\right)\omega_2^6 + 2\left(9\omega_3^6 + 57\omega_3^4 - 82\omega_3^2 + 240\right)\omega_2^4 - \\
&4\left(9\omega_3^6 + \omega_3^4 - 104\omega_3^2 - 144\right)\omega_2^2 - 64\omega_3^2\left(3\omega_3^2 + 4\right), \\
t_{14} &= 12\left(\omega_3^2 + 1\right)\omega_2^{14} + \left(-26\omega_3^4 - 135\omega_3^2 + 48\right)\omega_2^{12} + \\
&\left(16\omega_3^6 + 192\omega_3^4 + 121\omega_3^2 - 104\right)\omega_2^{10} - \left(2\omega_3^8 + 86\omega_3^6 + 157\omega_3^4 - 516\omega_3^2 + 336\right)\omega_2^8 + \\
&\omega_3^2\left(11\omega_3^6 + 106\omega_3^4 - 843\omega_3^2 - 391\right)\omega_2^6 + \\
&\left(-24\omega_3^8 + 161\omega_3^6 + 1525\omega_3^4 + 752\omega_3^2 - 832\right)\omega_2^4 - \\
&4\left(\omega_3^8 + 148\omega_3^6 + 484\omega_3^4 + 141\omega_3^2 + 48\right)\omega_2^2 + 64\omega_3^2\left(\omega_3^6 + 8\omega_3^4 + 12\omega_3^2 + 4\right), \\
t_{15} &= -4\left(3\omega_3^2 + 1\right)\omega_2^{14} + \left(25\omega_3^4 + 221\omega_3^2 - 60\right)\omega_2^{12} - \\
&\left(14\omega_3^6 + 319\omega_3^4 + 641\omega_3^2 - 144\right)\omega_2^{10} + \left(\omega_3^8 + 119\omega_3^6 + 1054\omega_3^4 - 592\omega_3^2 - 128\right)\omega_2^8 + \\
&\left(-9\omega_3^8 - 361\omega_3^6 - 608\omega_3^4 + 3088\omega_3^2 + 768\right)\omega_2^6 + \\
&4\left(9\omega_3^8 + 121\omega_3^6 - 469\omega_3^4 - 865\omega_3^2 + 864\right)\omega_2^4 - \\
&16\left(4\omega_3^8 - 9\omega_3^6 - 207\omega_3^4 - 298\omega_3^2 - 144\right)\omega_2^2 - 256\omega_3^2\left(3\omega_3^4 + 10\omega_3^2 + 6\right), \\
t_{16} &= 4\left(\omega_3^2 + 3\right)\omega_2^{14} + \left(-9\omega_3^4 - 39\omega_3^2 + 8\right)\omega_2^{12} + 2\left(3\omega_3^6 + 13\omega_3^4 - 2\omega_3^2 - 60\right)\omega_2^{10} + \\
&\left(-\omega_3^8 - 6\omega_3^6 + 122\omega_3^4 + 113\omega_3^2 + 72\right)\omega_2^8 + \left(\omega_3^8 - 60\omega_3^6 - 399\omega_3^4 + 74\omega_3^2 + 16\right)\omega_2^6 + \\
&\left(6\omega_3^8 + 199\omega_3^6 + 376\omega_3^4 - 41\omega_3^2 - 96\right)\omega_2^4 - \omega_3^2\left(29\omega_3^6 + 242\omega_3^4 + 365\omega_3^2 - 120\right)\omega_2^2 + \\
&16\left(2\omega_3^8 + 7\omega_3^6 + 6\omega_3^4 + \omega_3^2\right), \\
t_{17} &= 4\omega_2^{12} - \left(5\omega_3^2 + 8\right)\omega_2^{10} + \left(-3\omega_3^4 + 3\omega_3^2 - 8\right)\omega_2^8 + \left(5\omega_3^6 + 14\omega_3^4 + 16\omega_3^2 + 12\right)\omega_2^6 - \\
&\omega_3^2\left(\omega_3^6 + 11\omega_3^4 + 24\omega_3^2 - 6\right)\omega_2^4 + 2\omega_3^2\left(\omega_3^6 + 10\omega_3^4 + 7\omega_3^2 - 10\right)\omega_2^2 - 4\omega_3^4\left(\omega_3^2 + 1\right)^2, \\
t_{18} &= \omega_2^6 - 6\omega_2^4 + 6\omega_2^2 + 8, \\
t_{19} &= \omega_2^4 - \left(2\omega_3^2 + 1\right)\omega_2^2 + \omega_3^4 + \omega_3^2, \\
t_{20} &= \omega_2^2\omega_3^2 - 4, \\
t_{21} &= \left(\omega_2^2 - \omega_3^2\right)\left(-\omega_2^2 + \omega_3^2 + 1\right)^2,
\end{aligned}
$$

5.5.1 Analysis of quasi-convexity and Nekhoroshev stability estimates around P_3

We introduce the notation λ_1, λ_2 the eigenvalues of the matrix A_Λ, which is the submatrix of the rotation of A.

Theorem 5.1. *In the regions* $W_1 = W_{11} \cup W_{12} = \{(C, A) : \lambda_1(C, A) > 0, \lambda_2(C, A) > 0\}$ *and* $W_2 = W_{21} \cup W_{22} = \{(C, A) : \lambda_1(C, A) < 0, \lambda_2(C, A) < 0\}$, *the Hamiltonian* (5.6) *is quasi-convex. Then, for* ε *sufficiently small, any motion with initial conditions such that* $|I(0)| \leq \varepsilon$ *satisfies estimates*

$$|I(t)| \leq \varepsilon^{1/6}, \quad t \leq \exp\left(\varepsilon^{-1/6}\right),$$

as well as

$$|I(t)| \leq \varepsilon^{1/2}, \quad t \leq \exp\left(\varepsilon^{-1/6}\right).$$

Proof. We are going to analyze the quasi-convexity of (5.6). We denote by Λ the 2-dimensional linear space orthogonal to ω and by A the Hessian matrix of $\mathcal{H}(I)$ computed at the origin. We construct an orthonormal vector basis e_1, e_2, e_3 such that e_1 is parallel to ω, and perform a rotation of the coordinates I. We denote by R the rotation matrix.

$$R = \begin{pmatrix} r_{11} & -r_{21} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ -r_{13} & r_{23} & r_{33} \end{pmatrix} = \frac{1}{||\omega||} \begin{pmatrix} \omega_1 & -\omega_2 & \omega_3 \\ \omega_2 & \frac{\omega_1\omega_2^2 + \omega_3^2||\omega||}{\omega_2^2+\omega_3^2} & \frac{(||\omega||-\omega_1)\omega_2\omega_3}{\omega_2^2+\omega_3^2} \\ -\omega_3 & \frac{(||\omega||-\omega_1)\omega_2\omega_3}{\omega_2^2+\omega_3^2} & \frac{\omega_2^2||\omega|| + \omega_1\omega_3^2}{\omega_2^2+\omega_3^2} \end{pmatrix} \tag{5.7}$$

Then we take the appropriate 2×2 submatrix A_Λ of the rotation of A, which represents the restriction of the Hessian matrix to the space Λ. We compute the eigenvalues λ_1, λ_2 of A_Λ of the form

$$\lambda_{1,2} = -\frac{m_1 \pm \sqrt{m_2}}{4||\omega||^2},$$

with

$$\begin{aligned}
m_1 \;&= \omega_1^2 (A - 4a_{020}) + \omega_2^2 (A - 4a_{200}) + \omega_1 (4a_{101}\omega_3 - 4a_{110}\omega_2) - 4 (a_{020} + a_{200}) \omega_3^2 - \\
&4a_{011}\omega_2\omega_3, \\
m_2 \;&= 16 \left\{a_{011}^2 \left(\omega_1^2 + \omega_2^2\right) \left(\omega_1^2 + \omega_3^2\right) + a_{020}^2 \left(\omega_1^2 + \omega_3^2\right)^2 + \left(\omega_2^2 + \omega_3^2\right) \left(a_{101}^2 \left(\omega_1^2 + \omega_2^2\right) - \right.\right. \\
&2a_{200}a_{101}\omega_1\omega_3 + 2a_{110}\omega_2 (a_{200}\omega_1 + a_{101}\omega_3) + a_{110}^2 \left(\omega_1^2 + \omega_3^2\right) + a_{200}^2 \left(\omega_2^2 + \omega_3^2\right)) + \\
&2a_{011} \left(a_{101}\omega_1\omega_2 \left(\omega_1^2 + \omega_2^2\right) - \omega_3 \left(a_{110}\omega_1 \left(\omega_1^2 + \omega_3^2\right) + a_{200}\omega_2 \left(2\omega_1^2 + \omega_2^2 + \omega_3^2\right)\right)\right) + \\
&2a_{020} \left(a_{110}\omega_1\omega_2 \left(\omega_1^2 + \omega_3^2\right) + a_{200} \left(\omega_1^2 \left(\omega_2^2 - \omega_3^2\right) - \omega_3^2 \left(\omega_2^2 + \omega_3^2\right)\right) + \right. \\
&\left.\left.\omega_3 \left(a_{011}\omega_2 \left(\omega_1^2 + \omega_3^2\right) + a_{101}\omega_1 \left(\omega_1^2 + 2\omega_2^2 + \omega_3^2\right)\right)\right)\right\} + \\
&8A \left(a_{200}\omega_2^4 - a_{011}\omega_3\omega_2^3 + a_{200}\omega_1^2\omega_2^2 + a_{200}\omega_3^2\omega_2^2 + a_{101}\omega_1\omega_3\omega_2^2 - a_{011}\omega_1^2\omega_3\omega_2 + \right. \\
&a_{110}\omega_1 \left(\omega_1^2 + \omega_2^2 + 2\omega_3^2\right) \omega_2 - a_{200}\omega_1^2\omega_3^2 + a_{101}\omega_1^3\omega_3 + a_{020} \left(\omega_1^4 + \left(\omega_2^2 + \omega_3^2\right) \omega_1^2 - \right. \\
&\left.\left.\omega_2^2\omega_3^2\right)\right) + A^2 \left(\omega_1^2 + \omega_2^2\right)^2.
\end{aligned}$$

Since they have opposite signs in the blue region as shown in the Figure 5.5.1, $\mathcal{H}(I)$ is not quasi-convex at the origin in that region and have same signs in the red region, so $\mathcal{H}(I)$ is quasi-convex at the origin in that region.

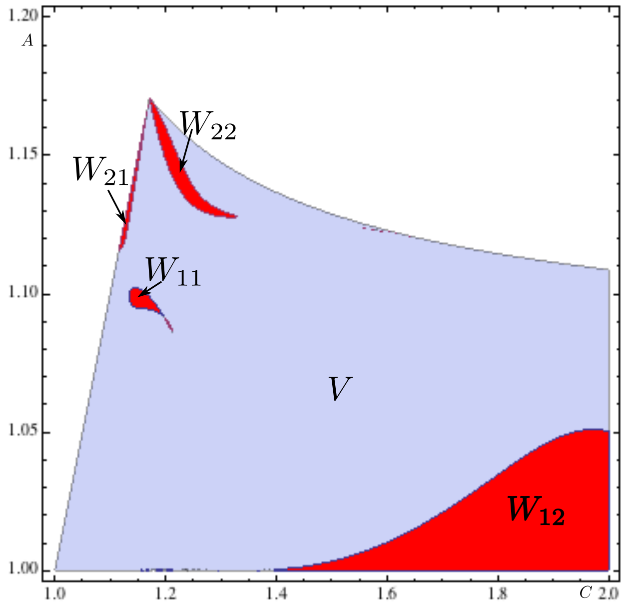


Figure 5.9: Regions where $\mathcal{H}(I)$ is quasi-convex (in red) and not quasi-convex (in blue).

∎

Observation 5.2. *According Theorem 5.1 of Nekhoroshev stability of P_3 cannot be decided by the use of Theorem 5.1 on the region $V \subset II$. In order to investigate the Nekhoroshev stability on V in the next section we are going to use Theorem 1.43 considering the condition of directionally quasi convexity.*

5.5.2 Analysis of directional quasi-convexity and Nekhoroshev stability estimates around P_3

We introduce the notation v_1, v_2 the eigenvectors of the matrix A_Λ, which is the submatrix of the rotation of A.

Theorem 5.3. *In the region $V_1 \cup V_2 \cup V_3 \cup V_4 \cup V_5 \cup V_6$, with $V_1 = \{(C, A) : v_1^A > 0, v_2^A > 0, v_3^A < 0\}$, $V_2 = \{(C, A) : v_1^A > 0, v_2^A < 0, v_3^A > 0\}$, $V_3 = \{(C, A) : v_1^B > 0, v_2^B > 0, v_3^B < 0\}$, $V_4 = \{(C, A) : v_1^B > 0, v_2^B < 0, v_3^B > 0\}$, $V_5 = \{(C, A) : v_1^B < 0, v_2^B < 0, v_3^B > 0\}$ and $V_6 = \{(C, A) : v_1^B < 0, v_2^B > 0, v_3^B < 0\}$ the Hamiltonian (5.6) is directionally quasi-convex. Then, for ε sufficiently small, any motion with initial conditions such that $|I(0)| \leq \varepsilon$ satisfies estimates*

$$|I(t)| \leq \varepsilon^{1/3}, \quad t \leq \exp\left(\varepsilon^{-1/3}\right),$$

as well as

$$|I(t)| \leq \varepsilon^{1/2}, \quad t \leq \exp\left(\varepsilon^{-1/6}\right).$$

Proof. The two eigenvalues of A_Λ are both nonvanishing, therefore, there are two directions in the space Λ on which the quadratic form $\mathcal{H}^2[v,v]$ vanishes, and they are generated by the unit vectors

$$\begin{aligned} v^A &= R^T\left(0, \sqrt{\frac{\lambda_2}{\lambda_2-\lambda_1}}x_1+\sqrt{\frac{-\lambda_1}{\lambda_2-\lambda_1}}y_1, \sqrt{\frac{\lambda_2}{\lambda_2-\lambda_1}}x_2+\sqrt{\frac{-\lambda_1}{\lambda_2-\lambda_1}}y_2\right),\\ v^B &= R^T\left(0, \sqrt{\frac{\lambda_2}{\lambda_2-\lambda_1}}x_1-\sqrt{\frac{-\lambda_1}{\lambda_2-\lambda_1}}y_1, \sqrt{\frac{\lambda_2}{\lambda_2-\lambda_1}}x_2-\sqrt{\frac{-\lambda_1}{\lambda_2-\lambda_1}}y_2\right),\end{aligned}$$

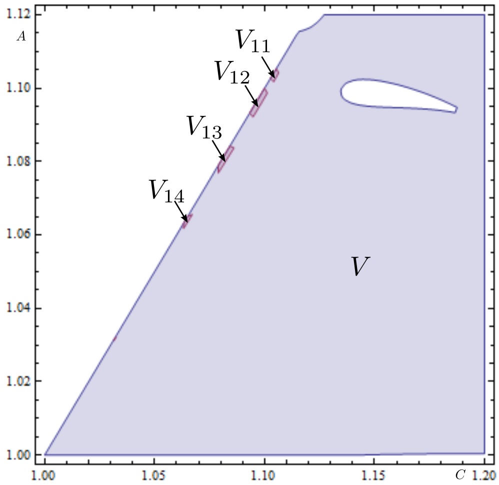
where $x=(x_1,x_2)=\frac{v_1}{||v_1||}$ and $y=(y_1,y_2)=\frac{v_2}{||v_2||}$ are the eigenvectors (of unitary norm) of the eigenvalues λ_1 and λ_2, respectively, namely:

$$v_{1,2}=\left(-\frac{4m_3+m_4\pm\sqrt{m_5}}{2m_6},1\right),$$

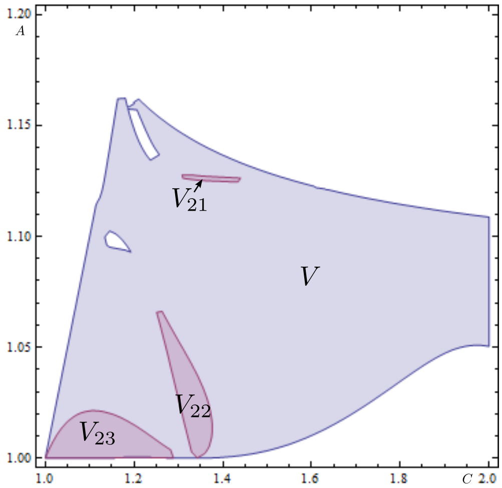
where

$$\begin{aligned}
m_3 &= a_{200}r_{13}^2-a_{110}r_{23}r_{13}-a_{101}r_{33}r_{13}+a_{020}r_{23}^2+a_{002}r_{33}^2+a_{011}r_{23}r_{33},\\
m_4 &= -2a_{200}r_{13}^2+2a_{110}r_{23}r_{13}+2a_{101}r_{33}r_{13}-2a_{200}r_{21}^2-2a_{020}r_{22}^2-2a_{002}r_{23}^2-\\
&\quad 2a_{020}r_{23}^2-2a_{002}r_{33}^2-2a_{110}r_{21}r_{22}-2a_{101}r_{21}r_{23}-2a_{011}r_{22}r_{23}-2a_{011}r_{23}r_{33},\\
m_5 &= \big(-2a_{200}r_{13}^2+2a_{110}r_{23}r_{13}+2a_{101}r_{33}r_{13}-2a_{200}r_{21}^2-2a_{020}r_{22}^2-2a_{002}r_{23}^2-\\
&\quad 2a_{020}r_{23}^2-2a_{002}r_{33}^2-2a_{110}r_{21}r_{22}-2a_{101}r_{21}r_{23}-2a_{011}r_{22}r_{23}-2a_{011}r_{23}r_{33}\big)^2-\\
&\quad 4\big(-a_{011}^2r_{23}^4+4a_{002}a_{020}r_{23}^4+2a_{011}a_{101}r_{13}r_{23}^3-4a_{002}a_{110}r_{13}r_{23}^3+4a_{020}a_{101}r_{21}r_{23}^3-\\
&\quad 2a_{011}a_{110}r_{21}r_{23}^3-a_{101}^2r_{13}^2r_{23}^2+4a_{002}a_{200}r_{13}^2r_{23}^2-a_{110}^2r_{21}^2r_{23}^2+4a_{020}a_{200}r_{21}^2r_{23}^2-\\
&\quad 2a_{101}a_{110}r_{13}r_{21}r_{23}^2+4a_{011}a_{200}r_{13}r_{21}r_{23}^2+\\
&\quad 4a_{020}a_{101}r_{13}r_{22}r_{23}^2-2a_{011}a_{110}r_{13}r_{22}r_{23}^2+2a_{011}a_{101}r_{21}r_{33}r_{23}^2-4a_{002}a_{110}r_{21}r_{33}r_{23}^2+\\
&\quad 2a_{011}^2r_{22}r_{33}r_{23}^2-8a_{002}a_{020}r_{22}r_{33}r_{23}^2-2a_{101}a_{110}r_{13}^2r_{22}r_{23}+4a_{011}a_{200}r_{13}^2r_{22}r_{23}-\\
&\quad 2a_{110}^2r_{13}r_{21}r_{22}r_{23}+8a_{020}a_{200}r_{13}r_{21}r_{22}r_{23}-2a_{101}a_{110}r_{21}^2r_{33}r_{23}+\\
&\quad 4a_{011}a_{200}r_{21}^2r_{33}r_{23}-2a_{101}^2r_{13}r_{21}r_{33}r_{23}+a_{002}a_{200}r_{13}r_{21}r_{33}r_{23}-\\
&\quad 2a_{011}a_{101}r_{13}r_{22}r_{33}r_{23}+4a_{002}a_{110}r_{13}r_{22}r_{33}r_{23}-4a_{020}a_{101}r_{21}r_{22}r_{33}r_{23}+\\
&\quad 2a_{011}a_{110}r_{21}r_{22}r_{33}r_{23}-a_{110}^2r_{13}^2r_{22}^2+4a_{020}a_{200}r_{13}^2r_{22}^2-a_{101}^2r_{21}^2r_{33}^2+\\
&\quad 4a_{002}a_{200}r_{21}^2r_{33}^2-a_{011}^2r_{22}^2r_{33}^2+4a_{002}a_{020}r_{22}^2r_{33}^2-2a_{011}a_{101}r_{21}r_{22}r_{33}^2+\\
&\quad 4a_{002}a_{110}r_{21}r_{22}r_{33}^2-4a_{020}a_{101}r_{13}r_{22}^2r_{33}+2a_{011}a_{110}r_{13}r_{22}^2r_{33}-\\
&\quad 2a_{101}a_{110}r_{13}r_{21}r_{22}r_{33}+4a_{011}a_{200}r_{13}r_{21}r_{22}r_{33}\big),\\
m_6 &= r_{23}\left(-a_{101}r_{13}+a_{011}r_{23}+2a_{002}r_{33}\right)+r_{22}\left(-a_{110}r_{13}+2a_{020}r_{23}+a_{011}r_{33}\right)+\\
&\quad r_{21}\left(-2a_{200}r_{13}+a_{110}r_{23}+a_{101}r_{33}\right)
\end{aligned}$$

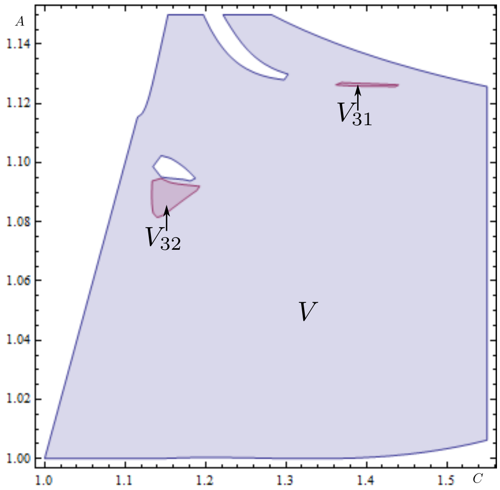
We computed the vectors v^A, v^B and since the vectors v^A and v^B does not have the components of the same sign in 12 cases, that are reduced to 6, because $v_1^A > 0$ in region II which are shown in the following Figure 5.10.



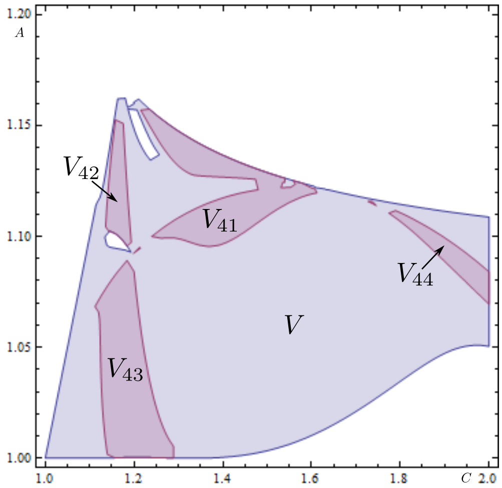
(a) $V_1 = \bigcup_{j=1}^{4} V_{1j} = \left\{(C, A) : v_1^A > 0, v_2^A > 0, v_3^A < 0\right\}$.



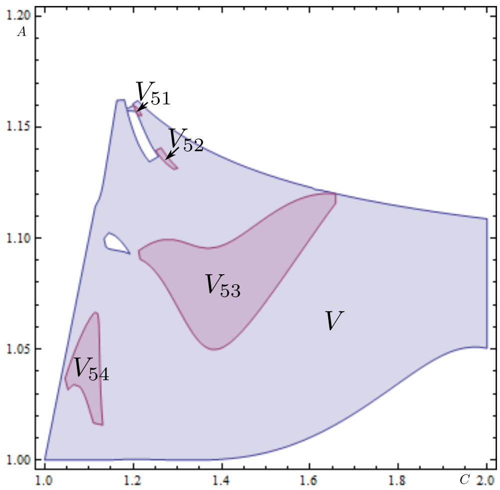
(b) $V_2 = \bigcup_{j=1}^{3} V_{2j} = \left\{(C, A) : v_1^A > 0, v_2^A < 0, v_3^A > 0\right\}$.



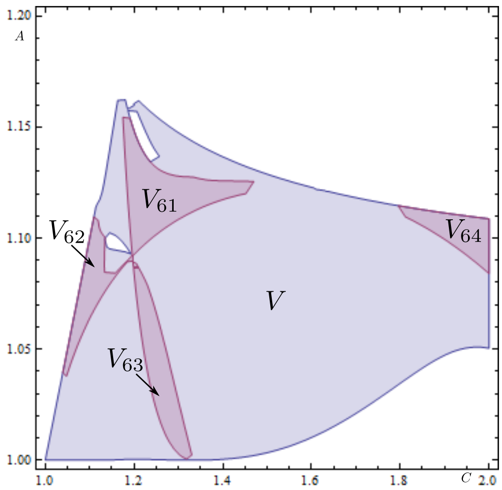
(c) $V_3 = \bigcup_{j=1}^{2} V_{3j} = \left\{(C, A) : v_1^B > 0, v_2^B > 0, v_3^B < 0\right\}$.



(d) $V_4 = \bigcup_{j=1}^{4} V_{4j} = \left\{(C, A) : v_1^B > 0, v_2^B < 0, v_3^B > 0\right\}$.



(e) $V_5 = \bigcup_{j=1}^{4} V_{5j} = \left\{(C, A) : v_1^B < 0, v_2^B < 0, v_3^B > 0\right\}$.



(f) $V_6 = \bigcup_{j=1}^{4} V_{6j} = \left\{(C, A) : v_1^B < 0, v_2^B > 0, v_3^B < 0\right\}$.

Figure 5.10: Regions $V_1, V_2, V_3, V_4, V_5, V_6$ where $\mathcal{H}(I)$ is directional quasi-convex.

■

Chapter 6

Conclusions and future work

The purpose of this work is the analysis on the nonlinear stability of elliptic equilibria in Hamiltonian systems with n degrees of freedom, where $n \geq 2$. The main achievements of this thesis are as follows:

1. Previous results on Lie stability have been generalized, providing a criterion which leads to formal stable systems under quite weak assumptions. In particular our result includes Nekhorosev stability of elliptic equilibria, but it also handles cases where stability is established from the normal form terms (of order two or higher) that can depend only on action coordinates or on a combination of some actions and resonant angles.

2. For the Lie stable systems we obtain time estimates of exponential type similar to those of the Nekhorosev theory. We need a Diophantine condition on the frequencies obtained after rewriting the quadratic part of the Hamiltonian in terms of the first integrals of motion related to the (truncated) normal form Hamiltonian.

3. We state two theorems on instability based on the construction of suitable Chetaev functions and the concept of an invariant ray. Our approach generalizes previous results on the same topic, dealing with interesting applications.

4. We have analysed the case of the triangular points L_4, L_5 for the spatial circular restricted three body problem applying our theory on Lie stability and instability. Our results enlarges previous results based either on Nekhoroshev stability, see [7, 39], or on formal stability, see [50].

5. The case of a satellite moving around its centre of mass and moving in a circular orbit is also considered, and in particular an equilibrium point of elliptic character. The problem depends on two parameters and we identify regions in the parameter plane where either Lie stability or instability holds. The analysis performed in [52] is generalized. Moreover we apply Nekhoroshev theory identifying the sets in the parametric plane where the equilibrium is directionally quasi convex, comparing this kind of stability with Lie stability.

6. For the two applications treated in the thesis we have bounded the solutions near the equilibrium over exponentially long times.

Currently two papers from this thesis have been published, both related with the instability analysis performed in this work. Specifically, Cárcamo, D.; Vidal, C.: Instability of equilibrium solutions of Hamiltonian systems with n-degrees of freedom under the existence of a single resonance and an invariant ray. *J. Differential Equations* **265** (2018) (12), 6295–6315 (reference [15]) and Cárcamo, D.; Vidal, C.: Instability of equilibrium solutions of Hamiltonian systems with n–degrees of freedom under the existence of multiple resonances and an application to the spatial satellite problem. *J. Dynamics and Differential Equations* (2018), https://doi.org/10.1007/s10884-018-9679-6 (reference [16]).

Furthermore there is a preprint about Lie stability and asymptotic estimates that has been submitted for publication recently, see [14]. D. Carcamo-Díaz, J.F. Palacián, C. Vidal, and P. Yanguas, Formal stability of elliptic equilibria in Hamiltonian systems with exponential time estimates, preprint.

As future work we point out two issues. The first one concerns the relationship between the existence of invariant tori of Lagrangian tori surrounding Lie stable elliptic equilibria. In this respect there are only a few results on this topic when $n > 2$. More specifically it is known that the condition on nonlinear stability is stronger than that about the existence of invariant tori, as it is possible to get examples of Hamiltonian systems that are unstable but such that the tori remain.

Another remaining topic is a deep comparison between the estimates of Nekhorosev type for elliptic equilibria and those provided in the present work. Since our development to get the bounds does not use Nekhorosev theory, it is clear that the results are different and can be compared only when one of the hypotheses that yield Nekhoroshev stability holds. It seems appropriate to perform this analysis from a numerical perspective and on specific examples, such as the equilibria studied in the thesis.

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